

Information Sources on a Bratteli Diagram

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Abstract

A Bratteli diagram is a type of graph in which the vertices are split into finite subsets occupying an infinite sequence of levels, starting with a bottom level and moving to successively higher levels along edges connecting consecutive levels. An information source on a Bratteli diagram consists of a sequence of PMFs on the vertex sets at each level that are compatible under edge transport. By imposing a regularity condition on the Bratteli diagram, we obtain various results for its information sources including ergodic and entropy rate decomposition theorems, a Shannon-McMillan-Breiman theorem, and lossless and lossy source coding theorems. Proof methodology exploits the Vershik transformation on the path space of a Bratteli diagram. Some results for finite alphabet stationary sequential information sources are seen to be a special case of the results of this paper.

1 Introduction

Let A be a finite set and let A^∞ be the product measurable space of one-sided sequences from A . The family of stationary sequential information sources with alphabet A can be viewed as the set $\mathcal{P}(A^\infty)$ of probability measures on A^∞ that are preserved by the shift transformation T_A on A^∞ . We also have the family of ergodic sources $\mathcal{P}_e(A^\infty)$, which consists of those sources in $\mathcal{P}(A^\infty)$ that are trivial on the sigma-field of T_A -invariant measurable subsets of A^∞ . Each source μ in $\mathcal{P}(A^\infty)$ has an entropy rate, defined as the entropy of the dynamical system (A^∞, T_A, μ) . In classical information source theory, there is a well-established body of results regarding the information sources in $\mathcal{P}(A^\infty)$. These results include the ergodic decomposition theorem [1], the entropy rate decomposition theorem [2] [3, Thm. 2.4.1], the Shannon-McMillan-Breiman theorem [4] [5], universal lossless source coding theorems [6], and a fixed-length lossy source coding theory [7].

Efforts have been expended in developing results analogous to the results just cited, for information sources going beyond the family of sequential sources $\mathcal{P}(A^\infty)$. For example, progress along these lines has occurred for asymptotically mean stationary information sources [8] [3, Chap. 4], for random fields on trees and on the lattice

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\mathbb{Z}^d for $d > 1$ [9][10], for certain quantum source families [11] [12], and for random graphical structures [13].

This paper is concerned with the further development of information source theory on graphs. The type of graph considered is called a Bratteli diagram, formally defined below. Originally, Bratteli diagrams were put forward [14] as a tool to resolve structural questions concerning certain operator algebras (AF-algebras). Subsequently, they were studied as a means to obtain models of measurable dynamical systems [15][16] and certain topological dynamical systems [17]. In recent years, there has been extensive work on ergodic-theoretic questions arising from a Bratteli diagram D ; the papers [18][19][20][21][22] are representative of this ergodic-theoretic line of research.

The vertices of a Bratteli diagram D occupy a countably infinite number of levels, namely, a level 0, a level 1, a level 2, etc. Edges of D can only connect vertices between consecutive levels. We see in Section II that there is a path space Ω_D associated with D consisting of certain infinite paths, each path starting at level 0 and ascending consecutively from level to level following edges connecting levels. There is also a natural one-to-one transformation T of Ω_D onto itself called the Vershik transformation. A Bratteli diagram D induces various dynamical systems called Bratteli-Vershik systems, which are the triples (Ω_D, T, P) in which P is a probability measure on Ω_D preserved by T . As we shall see, each Bratteli-Vershik system gives rise to a certain information source which we call a Bratteli-Vershik source. We shall obtain results for Bratteli-Vershik sources analogous to classical source coding theory results obtained for the sequential source family $\mathcal{P}(A^\infty)$. In analyzing a Bratteli-Vershik source, one exploits how the Vershik transformation T acts on the path space Ω_D , whereas, in analyzing a sequential source in $\mathcal{P}(A^\infty)$, one exploits how the shift transformation T_A acts on the sequence space A^∞ . Since the action of T on Ω_D is unlike the action of T_A on A^∞ , proof methodologies for obtaining Bratteli-Vershik source results differ from the methodologies for obtaining sequential source results, even though the two types of results are analogous.

Notation and Terminology.

- $|S|$ denotes the cardinality of finite set S , S^* denotes the set of all finite-length strings with entries from S , and $|s|$ denotes the length of string $s \in S^*$. If s_i is a string in S^* for $1 \leq i \leq n$, then $s_1 s_2 \cdots s_n \in S^*$ denotes the string which is the concatenation of s_1, s_2, \dots, s_n . S^* is a semigroup under the concatenation operation $(s_1, s_2) \rightarrow s_1 s_2$.
- The terminology *random object* designates a measurable function mapping a probability space into some measurable space; *random variable* designates a random object which takes its values in the real line (or extended real line).
- A finite set S is taken to be the measurable space with measurable sets all subsets of S , and if μ is a PMF on S , we shall also use μ to denote the probability measure on S induced by μ .
- The Shannon entropy $H(\mu)$ of a PMF μ on finite set S is defined by

$$H(\mu) \triangleq \sum_{s \in S, \mu(s) > 0} -\mu(s) \log_2 \mu(s).$$

Furthermore, if X is an S -valued random object on probability space $(\Omega, \mathcal{F}, \mathcal{P})$ whose PMF is μ , then $H(X) \triangleq H(\mu)$, or we write $H_P(X)$ for $H(X)$ if the probability measure P is to be emphasized. All elementary properties of entropy (such as concavity and subadditivity) are assumed throughout.

Definition: Bratteli diagram. Suppose we have a graph (V, E) such that

- **(a.1):** The set of vertices V is countably infinite and decomposes as a disjoint union $V = \bigcup_{n=0}^{\infty} V_n$ in which each V_n is finite and nonempty. V_n is referred to as the set of vertices of V at level n .
- **(a.2):** The set of edges E is countably infinite and decomposes as a disjoint union $E = \bigcup_{n=0}^{\infty} E_n$ in which each E_n is finite and nonempty.
- **(a.3):** For each $n \geq 0$, each edge e in E_n connects a vertex in V_n (called the *source vertex* of e and denoted $s(e)$) with a vertex in V_{n+1} (called the *range vertex* of e and denoted $r(e)$). Thus, the edges in E_n connect level n vertices with level $n+1$ vertices.
- **(a.4):** For each $n \geq 0$ and each vertex $v \in V_n$, there is at least one edge $e \in E_n$ such that $s(e) = v$.
- **(a.5):** For each $n \geq 1$ and each vertex $v \in V_n$, the set of edges $E(v) \triangleq \{e \in E_{n-1} : r(e) = v\}$ is nonempty.

Then $D = (V, E)$ is called a *Bratteli diagram*. Two Bratteli diagrams are said to be isomorphic if they are isomorphic as graphs.

Figures 1(a) and 1(b) illustrate two examples of Bratteli diagrams discussed in the following subsection.

1.1 Bratteli diagrams via multisets

The multiset concept is a generalization of the concept of set and is a useful concept in dealing with Bratteli diagrams. Recall that a set is an unordered collection of distinct elements. A multiset is an unordered collection of elements in which repetitions of elements are allowed. Each distinct element s of a multiset M appears in M with a certain multiplicity $m(s|M)$. The cardinality $|M|$ of M is the total number of elements of M including multiplicities, that is, $|M|$ is the sum of $m(s|M)$ over the distinct elements s of M . For example, for the multiset $M = \{a, a, a, b, b\}$, we have $m(a|M) = 3$, $m(b|M) = 2$, and $|M| = m(a|M) + m(b|M) = 5$.

Let $D = (V, E)$ be a Bratteli diagram. V^+ denotes the set of vertices $\bigcup_{n \geq 1} V_n$. $\{E(v) : v \in V^+\}$ forms a partition of the edge set E . For each $v \in V^+$, we define $M_D(v)$ to be the multiset of cardinality $|E(v)|$ whose set of distinct elements is $\{s(e) : e \in E(v)\}$ and the multiplicity of each vertex v' in this set is

$$m(v'|M_D(v)) = |\{e \in E(v) : s(e) = v'\}|.$$

Suppose that one knows the levels $\{V_n : n \geq 0\}$ of the vertex set of a Bratteli diagram $D = (V, E)$ as well as the multisets $\{M_D(v) : v \in V^+\}$. Then D is uniquely determined

up to isomorphism. Consequently, we sometimes specify a Bratteli diagram $D = (V, E)$ by specifying its vertex set levels $\{V_n\}$ and the multisets associated with the vertices in V^+ . We do this in Examples 1.1 and 1.2 which follow.

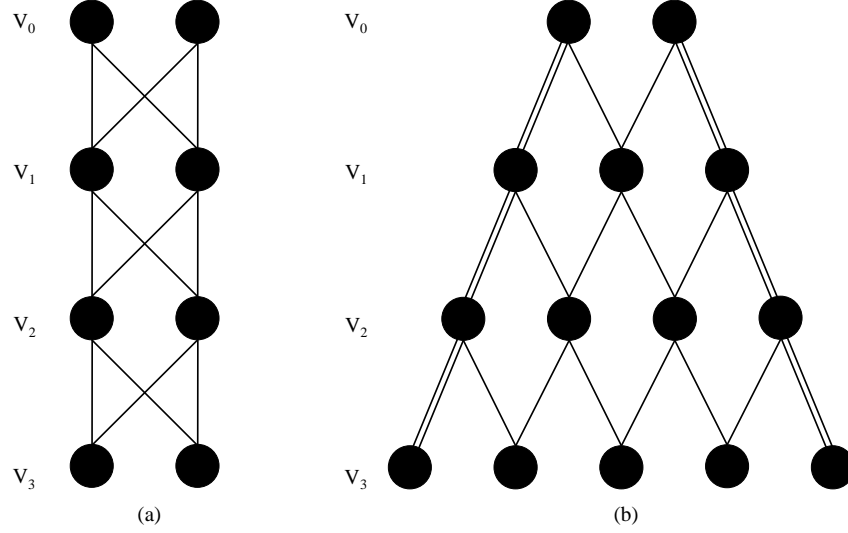


Figure 1: Levels 0-3 of regular Bratteli diagrams in Ex. 1.1 (a) and Ex. 1.2 (b).

Example 1.1. Consider the Bratteli diagram $D = (V, E)$ in which

$$V_n = \{v_0(n), v_1(n)\}, \quad n \geq 0,$$

$$M_D(v) = \{v_0(n-1), v_1(n-1)\}, \quad v \in V_n, \quad n \geq 1.$$

Figure 1(a) depicts levels 0-3 of D (consisting of V_0 through V_3 and their connecting edges).

Example 1.2. Let $D = (V, E)$ be the Bratteli diagram such that

$$V_n = \{v_0(n), v_1(n), \dots, v_{n+1}(n)\}, \quad n \geq 0$$

and such that

$$\begin{aligned} M_D(v_0(n)) &= \{v_0(n-1), v_0(n-1)\}, \\ M_D(v_i(n)) &= \{v_{i-1}(n-1), v_i(n-1)\}, \quad 0 < i < n+1 \\ M_D(v_{n+1}(n)) &= \{v_n(n-1), v_n(n-1)\} \end{aligned}$$

for $n \geq 1$. Levels 0-3 of D are depicted in Figure 1(b), from which it is seen that D is based on Pascal's triangle in a natural way.

1.2 Bratteli-Vershik information sources

Let $D = (V, E)$ be a Bratteli diagram. Suppose $n \geq 0$ and λ is a PMF on V_{n+1} . We perform the following two-step random experiment which creates a $(V_{n+1} \times V_n)$ -valued random pair (X_{n+1}, X_n) .

- In Step 1, random vertex $X_{n+1} \in V_{n+1}$ is selected whose PMF is λ .
- In Step 2, given $X_{n+1} = v$, an edge is selected randomly from $E(v)$ (according to the equiprobable distribution on $E(v)$) and then random $X_n \in V_n$ is the source vertex of this edge.

Let $[\lambda]$ denote the PMF of X_n on V_n . It is easily worked out that

$$[\lambda](v) = \sum_{e \in E_n: s(e)=v} |E(r(e))|^{-1} \lambda(r(e)), \quad v \in V_n.$$

In this way, the edges in E_n are used to transport PMF λ on V_{n+1} into PMF $[\lambda]$ on V_n . We define $\mathcal{S}(D)$ to be the set of all functions $\mu : V \rightarrow [0, 1]$ such that the restriction μ_n of μ to V_n is a PMF ($n \geq 0$) and

$$\mu_n = [\mu_{n+1}], \quad n \geq 0.$$

The preceding equation constitutes a consistency relationship among the set of PMFs $\{\mu_n : n \geq 0\}$ comprising μ , vis-a-vis the diagram D . The members of $\mathcal{S}(D)$ are called the *Bratteli-Vershik information sources on D* (B-V sources on D). It is straightforward to show that $\mathcal{S}(D)$ is a convex set, that is, if μ^1, μ^2 are any pair of distinct elements of $\mathcal{S}(D)$,

$$S(\mu^1, \mu^2) \triangleq \{\alpha\mu^1 + (1-\alpha)\mu^2 : \alpha \in (0, 1)\} \subset \mathcal{S}(D).$$

The extreme points of $\mathcal{S}(D)$ are the elements of $\mathcal{S}(D)$ not belonging to any of the sets $S(\mu^1, \mu^2)$. We let $\mathcal{S}_e(D)$ denote the set of all extreme points of $\mathcal{S}(D)$. The members of $\mathcal{S}_e(D)$ are the *ergodic Bratteli-Vershik information sources on D* (ergodic B-V sources on D).

We point out some examples of B-V sources. For the Bratteli diagram $D = (V, E)$ of Example 1.1, we have $\mathcal{S}(D) = \mathcal{S}_e(D)$, each consisting of the unique Bratteli-Vershik source $\mu : V \rightarrow [0, 1]$ in which $\mu(v) = 1/2$ for every $v \in V$. For the Bratteli diagram $D = (V, E)$ of Example 1.2, it can be shown that there are exactly two ergodic Bratteli-Vershik sources, namely, the source $\sigma : V \rightarrow [0, 1]$ for which

$$\sigma(v_0(n)) = 1, \quad n \geq 0,$$

and the source $\tau : V \rightarrow [0, 1]$ for which

$$\tau(v_{n+1}(n)) = 1, \quad n \geq 0.$$

$\mathcal{S}(D)$ is then the convex hull of $\{\sigma, \tau\}$. Examples 1.3-1.4 presented later on give Bratteli diagrams for which there are uncountably many ergodic B-V sources.

Topological Spaces $\mathcal{S}(D)$ and $\mathcal{S}_e(D)$. The source families $\mathcal{S}(D)$ and $\mathcal{S}_e(D)$ are each topological spaces with the topologies they inherit from the Cartesian product topology

on $[0, 1]^V$. $\mathcal{S}_e(D)$ is also a measurable space whose measurable sets are the Borel sets with respect to the topology on $\mathcal{S}_e(D)$. For each $v \in V$, the mapping $\sigma \rightarrow \sigma(v)$ from $\mathcal{S}_e(D)$ into the real line is continuous and therefore is a Borel measurable mapping.

Lemma 1.1. Let $D = (V, E)$ be any Bratteli diagram. Then:

- $\mathcal{S}(D)$ is non-empty and compact.
- $\mathcal{S}_e(D)$ is a countable intersection of open subsets of $\mathcal{S}(D)$.
- For each $\mu \in \mathcal{S}(D)$, there exists a probability measure λ_μ on $\mathcal{S}_e(D)$ such that

$$\mu(v) = \int_{\mathcal{S}_e(D)} \sigma(v) d\lambda_\mu(\sigma), \quad v \in V.$$

Proof. Let Bratteli diagram $D = (V, E)$ be given. \mathbb{R}^V is the vector space of all real-valued mappings on V . \mathbb{R}^V is a locally convex topological vector space under the Cartesian product topology. $[0, 1]^V$ is a compact subset of \mathbb{R}^V . For each $N \geq 1$, let $\mathcal{S}_N(D)$ be the set of all $\mu \in [0, 1]^V$ such that the restriction μ_n of μ to V_n is a PMF ($n \geq 0$) and $\mu_n = [\mu_{n+1}]_D$ holds for $0 \leq n < N$. Each $\mathcal{S}_N(D)$ is non-empty and is a closed and therefore a compact subset of $[0, 1]^V$. Consequently, $\mathcal{S}(D)$ is also non-empty and compact because it is the intersection of the $\mathcal{S}_N(D)$'s and the sequence of sets $\{\mathcal{S}_N(D) : N \geq 1\}$ is a nested sequence. The rest of Lemma 1.1 follows by applying the metrizable form of Choquet's theorem [23, p. 14] to the compact convex metrizable topological space $\mathcal{S}(D)$.

Remark. The *ergodic decomposition theorem* is said to hold for Bratteli diagram D if the probability measure λ_μ on $\mathcal{S}_e(D)$ in Lemma 1.1 is unique for each B-V source μ on D . In the following subsection, we introduce types of Bratteli diagrams for which the ergodic decomposition theorem holds.

1.3 Some Types of Bratteli Diagrams

Let β be an integer ≥ 2 . Let S be a finite set and let x be a string in the semigroup S^* such that $|x|/\beta$ is an integer. We define the β -decomposition of x to be the β -tuple $(x[0], x[1], \dots, x[\beta-1])$ in which each entry $x[i]$ is a string in S^* of length $|x|/\beta$ and the factorization

$$x = x[0]x[1] \cdots x[\beta-1]$$

holds.

Canonical Bratteli Diagrams. Let β be an integer ≥ 2 . A Bratteli diagram $D = (V, E)$ is said to be β -canonical if

- $|E(v)| = \beta$ for each $v \in V^+$.
- $V_n \subset V_0^{\beta^n}$ for each $n \geq 1$.
- For each $x \in V^+$, $M_D(x) = \{x[0], x[1], \dots, x[\beta-1]\}$, where $(x[0], x[1], \dots, x[\beta-1])$ is the β -decomposition of x .

A Bratteli diagram is said to be canonical if it is β -canonical for some $\beta \geq 2$.

Example 1.3. Let A be a finite non-empty set and let $\beta \geq 2$. Then we have the β -canonical Bratteli diagram $D_\beta(A) = (V, E)$ in which $V_n = A^{\beta^n}$ for $n \geq 0$. For each sequential source $\alpha \in \mathcal{P}(A^\infty)$, let $\alpha^\dagger : V \rightarrow [0, 1]$ be the mapping whose restriction to V_n is the marginal PMF of α on A^{β^n} ($n \geq 0$). It is easy to show that α^\dagger is a Bratteli-Vershik source on D . The mapping $\alpha \rightarrow \alpha^\dagger$ is thus an embedding (i.e., a one-to-one mapping) of $\mathcal{P}(A^\infty)$ into $\mathcal{S}(D_\beta(A))$. In Sec. II, we make use of this embedding to show that some universal source coding theorems for the sequential source family $\mathcal{P}(A^\infty)$ are derivable from universal source coding theorems for Bratteli-Vershik source family $\mathcal{S}(D_\beta(A))$. In this way, we can view some parts of sequential source coding theory as a special case of B-V source coding theory.

Regular Bratteli Diagrams. Let β be an integer ≥ 2 . A Bratteli diagram is defined to be β -regular if it is isomorphic to a β -canonical Bratteli diagram. A Bratteli diagram is regular if it is a β -regular diagram for some $\beta \geq 2$. A β -canonical Bratteli diagram is β -regular because it is isomorphic to itself. We develop a necessary and sufficient condition for a Bratteli diagram to be regular, by which we can easily determine whether or not a non-canonical diagram is regular. If M is a multiset, we let $N(M)$ be the number of distinct orderings of the elements of M . For example $N(\{a, a, b\}) = 3$ since we have the three orderings aab, aba, baa . If $S(M)$ is the set consisting of the distinct elements of multiset M , then

$$N(M) = \frac{|M|!}{\prod_{s \in S(M)} m(s|M)!}.$$

Lemma 1.2. Let β be an integer ≥ 2 . A Bratteli diagram $D = (V, E)$ is β -regular if and only if

- **(b.1):** $|E(v)| = \beta$ for every $v \in V^+$.
- **(b.2):** For any multiset $M \in \{M_D(v) : v \in V^+\}$,

$$|\{v \in V^+ : M_D(v) = M\}| \leq N(M).$$

Remarks. The diagram in Example 1.4 is regular because it is canonical. The diagrams in Examples 1.1-1.3 all satisfy (b.1)-(b.2) with $\beta = 2$, and thus these diagrams are also regular by Lemma 1.2.

Conditions (b.1)-(b.2) are invariant under isomorphism, and it is easy to see that these conditions hold for any β -canonical Bratteli diagram. Thus, (b.1)-(b.2) hold for any β -regular Bratteli diagram. The more interesting part of proving Lemma 1.2, accomplished next, is to show that properties (b.1)-(b.2) imposed on a Bratteli diagram D imply that D is isomorphic to a canonical diagram.

Isomorphisms via Indexings. Let β be an integer ≥ 2 . Let $D = (V, E)$ be a Bratteli diagram satisfying conditions (b.1)-(b.2). An *indexing* of D is any function $I : E \rightarrow \{0, 1, \dots, \beta - 1\}$ satisfying the following two properties:

- **(c.1):** $I(E(v)) = \{0, 1, \dots, \beta - 1\}$ for $v \in V^+$.
- **(c.2):** If v, v' are distinct vertices in V^+ , there exist edges $e \in E(v)$ and $e' \in E(v')$ such that $I(e) = I(e')$ and $s(e) \neq s(e')$.

Indexings of D exist because of property (b.2); Figure 2 illustrates indexings of the Bratteli diagrams in Examples 1.1-1.2. Fix any indexing I of D . We show how to use I to construct an isomorphism between D and some β -canonical Bratteli diagram $D' = (V', E')$. Such an isomorphism will consist of a pair of mappings (η, τ) in which η is a one-to-one mapping of V onto V' , τ is a one-to-one mapping of E onto E' , and the property

$$s(\tau(e)) = \eta(s(e)), \quad r(\tau(e)) = \eta(r(e)), \quad e \in E \quad (1)$$

holds. Let $n \geq 1$. For each $v \in V_n$, let G_v be the context-free grammar with set of non-terminal symbols $V_1 \cup V_2 \cup \dots \cup V_n$, set of terminal symbols V_0 , start symbol v , and production rules of the form

$$u \rightarrow (s(e_0(u)), s(e_1(u)), \dots, s(e_{\beta-1}(u))),$$

where u is a non-terminal symbol and $e_i(u)$ is the edge $e \in E(u)$ for which $I(e) = i$. The language $L(G_v)$ of G_v is a singleton set $L(G_v) = \{x(G_v)\}$, and $x(G_v)$ belongs to the set $V_0^{\beta^n}$. Let $D' = (V', E')$ be the β -canonical grammar in which

$$V'_0 = V_0,$$

$$V'_n = \{x(G_v) : v \in V_n\}, \quad n \geq 1,$$

$$E'(x) = \{e'_0(x), e'_1(x), \dots, e'_{\beta-1}(x)\}, \quad x \in (V')^+,$$

where

$$s(e'_i(x)) = x[i], \quad i = 0, 1, \dots, \beta - 1.$$

Let $\eta : V \rightarrow V'$ be the mapping

$$\eta(v) = v, \quad v \in V_0,$$

$$\eta(v) = x(G_v), \quad v \in V^+.$$

and let $\tau : E \rightarrow E'$ be the mapping

$$\tau(e_i(v)) = e'_i(\eta(v)), \quad v \in V^+, \quad i = 0, 1, \dots, \beta - 1.$$

Both of these mappings are one-to-one and onto. Because of the property

$$\eta(v) = \eta(s(e_0(v)))\eta(s(e_1(v))) \cdots \eta(s(e_{\beta-1}(v))), \quad v \in V^+,$$

(1) holds and thus (η, τ) is an isomorphism between D and D' . We will refer to the function η constructed above as η_I to denote its dependence upon indexing I .

For example, if $D = (V, E)$ is the 2-regular Bratteli diagram of Ex. 1.2, if we label the vertices depicted in Fig. 1(b) left-to-right as

$$V_0 = \{a, b\}, \quad V_1 = \{c, d, e\}, \quad V_2 = \{f, g, h, i\}, \quad V_3 = \{j, k, l, m, n\},$$

and if I is the indexing of D in Fig. 2, then we compute $\eta_I(l)$ via three rounds of substitutions as follows:

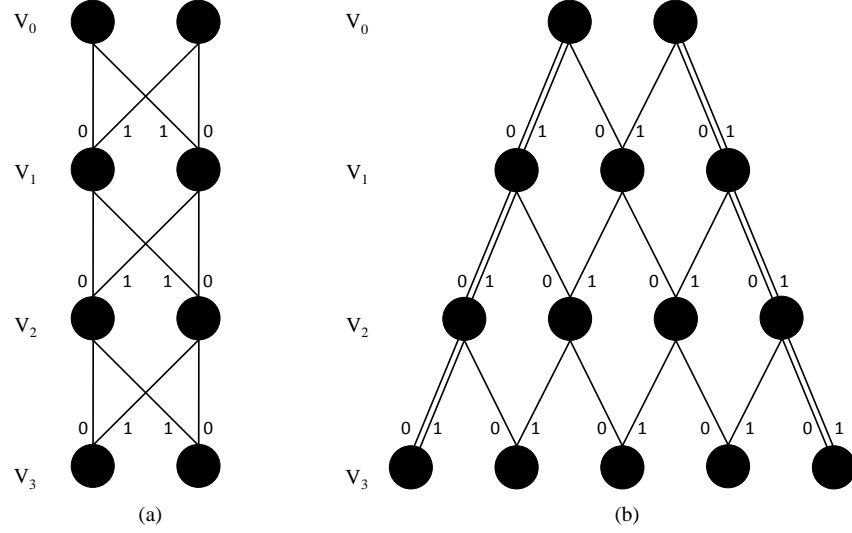


Figure 2: Indexings of diagrams of Ex. 1.1 and Ex. 1.2.

$$\begin{aligned}
& l \\
& \downarrow \\
& gh \\
& \downarrow \\
& cdde \\
& \downarrow \\
& \eta_l(l) = aaababbb
\end{aligned}$$

Background on Simplicial Grids. A simplicial grid in a Euclidean space \mathbb{E} is a sequence $\{K_n : n \geq 0\}$ in which (a) K_n is a finite simplicial complex of simplexes in \mathbb{E} for $n \geq 0$ and (b) K_n is a subdivision of K_{n-1} for $n \geq 1$. As we shall see below, certain simplicial grids in Euclidean spaces induce regular Bratteli diagrams in a natural way, but first we present some background on simplicial grids. Given simplicial grid $\{K_n : n \geq 0\}$ in \mathbb{E} , its underlying space is the compact subset Θ of \mathbb{E} such that

$$\Theta = \bigcup_{\sigma \in K_n} \sigma, \quad n \geq 0.$$

For each $n \geq 0$, the vertex set $V(K_n)$ of K_n is the set of all points x such that x is a vertex of some simplex in K_n . For each $x \in \Theta$ and $n \geq 0$, there is a unique PMF $p_{x|K_n}$ on $V(K_n)$ such that

$$x = \sum_{v \in V(K_n)} v p_{x|K_n}(v)$$

and $\{v \in V(K_n) : p_{x|K_n}(v) > 0\}$ is the set of vertices of a simplex in K_n . $p_{x|K_n}$ is called the *barycentric distribution* of x with respect to K_n .

Regular Diagrams Induced by Grids. Let β be an integer ≥ 2 . Simplicial grid $\{K_n : n \geq 0\}$ in Euclidean space \mathbb{E} is defined to be β -admissible if for each $n \geq 1$ and $x \in V(K_n)$, the barycentric distribution of x with respect to K_{n-1} takes its values in the set $\beta^{-1}\{0, 1, 2, \dots, \beta\}$, which allows us to define the unique multiset

$$M(x) \triangleq \{x_1, x_2, \dots, x_\beta\}$$

whose distinct entries form the vertex set of some simplex in K_{n-1} , and

$$x = \beta^{-1}(x_1 + x_2 + \dots + x_\beta).$$

A β -admissible simplicial grid $\{K_n : n \geq 0\}$ induces the β -regular Bratteli diagram $D = (V, E)$ in which

- **(d.1):** For $n \geq 0$,

$$V_n \triangleq V(K_n) \times \{n\} = \{(x, n) : x \in V(K_n)\}.$$

- **(d.2):** For $(x, n) \in V_n$ and $n \geq 1$,

$$\begin{aligned} M_D((x, n)) &\triangleq M(x) \times \{n-1\} \\ &= \{(x_1, n-1), (x_2, n-1), \dots, (x_\beta, n-1)\} \end{aligned}$$

If Θ is the underlying space of $\{K_n\}$, then for each $\theta \in \Theta$ define $\mu^\theta : V \rightarrow [0, 1]$ to be the mapping

$$\mu^\theta((x, n)) \triangleq p_{\theta|K_n}(x), \quad x \in V(K_n), \quad n \geq 0.$$

It can be shown that

$$\{p^\theta : \theta \in \Theta\} = S_e(D).$$

Example 1.4. We illustrate the Bratteli diagram induced by a well known simplicial grid called the Kuhn grid [24][25]. Fix any integer $\beta \geq 2$. Let \mathbb{E} be the Euclidean space \mathbb{R}^k , where $k \geq 1$. Let $S \subset \mathbb{E}$ be the hypercube

$$S = \{(x_1, x_2, \dots, x_k) : 0 \leq x_i \leq 1, \quad i = 1, 2, \dots, k\}.$$

Let Π_k be the set of all permutations of $\{1, 2, \dots, k\}$. For each $\pi \in \Pi_k$, let σ_π be the simplex

$$\sigma_\pi \triangleq \{(x_1, \dots, x_k) \in S : x_{\pi(1)} \leq x_{\pi(2)} \leq \dots \leq x_{\pi(k)}\}.$$

The set K_0 consisting of the $k!$ simplexes σ_π and their faces is a simplicial complex. For each $n \geq 1$, let K_n be the simplicial complex consisting of all simplexes of form

$$\beta^{-n}(\sigma_\pi + z), \quad \pi \in \Pi_k; z \in \mathbb{Z}^k,$$

that are subsets of S , together with all their faces. $\{K_n : n \geq 0\}$ is the (β, k) Kuhn grid, whose underlying space is the hypercube S . For each $n \geq 0$, the vertex set $V(K_n)$ of

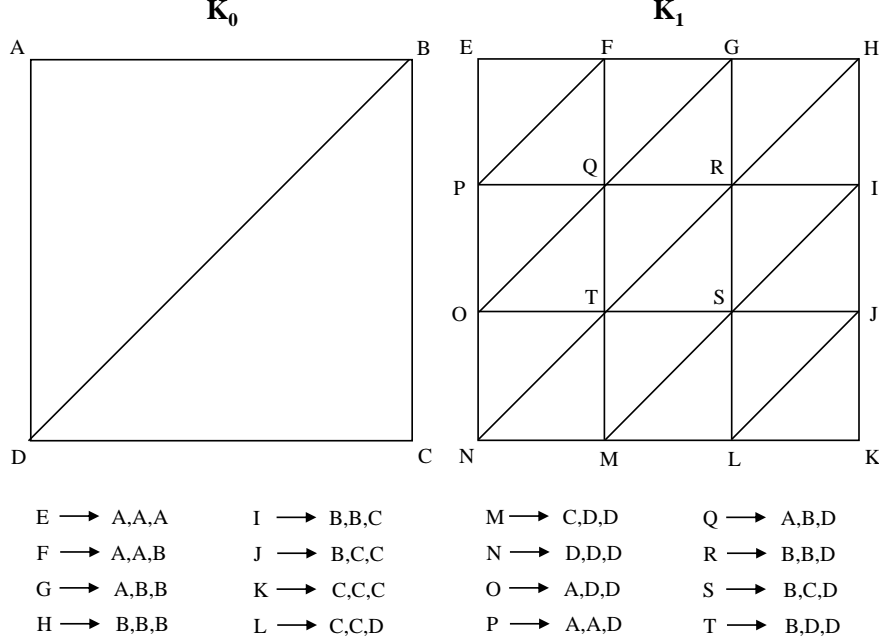


Figure 3: Above: Complexes K_0, K_1 of the $(\beta, k) = (3, 2)$ Kuhn grid $\{K_n : n \geq 0\}$ of Ex. 1.4. Below: Subgraph $(V_0 \cup V_1, E_0)$ of the induced β -regular Bratteli diagram (V, E) .

K_n consists of the $(\beta^n + 1)^k$ points in S of the form $(i_1, i_2, \dots, i_k)/\beta^n$ in which each $i_j \in \{0, 1, \dots, \beta^n\}$. Once the multisets $\{M(x) : x \in V(K_1)\}$ are determined, then each multiset $M(u)$ for $u \in V(K_n)$ and $n \geq 2$ can be determined by translation and scaling as follows: pick $z \in \mathbb{Z}^k$ such that

$$u \in \beta^{n-1}(S + z) \subset S;$$

then $x = \beta^{n-1}u - z$ belongs to $V(K_1)$ and

$$M(u) = \beta^{n-1}(M(x) + z).$$

The β -regular Bratteli diagram $D = (V, E)$ induced by the Kuhn grid is now determined according to (d.1)-(d.2). For the $(\beta, k) = (3, 2)$ Kuhn grid, Fig. 3 illustrates the subgraph $(V_0 \cup V_1, E_0)$ of the induced diagram $D = (V, E)$, where, for example, $I \rightarrow B, B, C$ in Fig. 3 means that $M_D(I) = \{B, B, C\}$ and consequently the three edges in $E(I)$ have source vertices B, B, C , respectively.

1.4 B-V Source Entropy Rate and its Interpretation

Lemma 1.4. Let $D = (V, E)$ be a β -regular Bratteli diagram, where $\beta \geq 2$. If $n \geq 0$ and λ is a PMF on V_{n+1} , then

$$H(\lambda) \leq \beta H([\lambda]). \quad (2)$$

Proof. Via Lemma 1.2, it suffices to prove the result assuming that D is canonical. Let X be a V_{n+1} -valued random object whose PMF is λ . Since X and its β -decomposition $(X[0], X[1], \dots, X[\beta-1])$ are functions of each other,

$$H(\lambda) = H(X) = H(X[0], X[1], \dots, X[\beta-1]).$$

By sub-additivity of the entropy function,

$$H(X[0], X[1], \dots, X[\beta-1]) \leq \sum_{i=0}^{\beta-1} H(X[i]).$$

Each $X[i]$ is V_n -valued. Let p_i be the PMF of $X[i]$ on V_n . We have

$$[\lambda] = \beta^{-1} \sum_{i=0}^{\beta-1} p_i.$$

By concavity of the entropy function and the preceding relations,

$$H([\lambda]) \geq \beta^{-1} \sum_{i=0}^{\beta-1} H(p_i) \geq \beta^{-1} H(\lambda),$$

and our result is proved.

Definition: Entropy Rate of a Bratteli-Vershik Source. Let $\beta \geq 2$, let D be any β -regular Bratteli diagram, and let μ be any source in $\mathcal{S}(D)$. By Lemma 1.4, the sequence $\{\beta^{-n} H(\mu_n) : n \geq 0\}$ is non-increasing and therefore it possesses a limit as $n \rightarrow \infty$. This limit is defined as the *entropy rate* $H_\infty(\mu)$ of μ , that is,

$$H_\infty(\mu) \triangleq \lim_{n \rightarrow \infty} \beta^{-n} H(\mu_n) = \inf_n \beta^{-n} H(\mu_n).$$

We have the upper bounds

$$H_\infty(\mu) \leq H(\mu_0) \leq \log_2 |V_0|, \quad \mu \in \mathcal{S}(D).$$

Coding Interpretation of Entropy Rate. We discuss the operational significance of entropy rate in the lossless encoding of Bratteli-Vershik sources. If S is a finite set, a one-to-one function $\phi : S \rightarrow \{0, 1\}^*$ satisfying the prefix condition is called a *lossless prefix encoder* on S . (The prefix condition means that the codeword $\phi(s_1)$ is not a prefix of codeword $\phi(s_2)$ for any two distinct s_1, s_2 in S .) Let p be any PMF on S . For any lossless prefix encoder ϕ on S , define

$$\bar{L}(\phi, p) \triangleq \sum_{s \in S} |\phi(s)| p(s),$$

the expected codeword length resulting from using ϕ to encode the members of S distributed according to p . Basic information theory gives us the following facts (e.1)-(e.2).

- **(e.1):** For any lossless prefix encoder ϕ on S and any PMF p on S ,

$$\bar{L}(\phi, p) \geq H(p).$$

- **(e.2):** For any PMF p on S , there exists a lossless prefix encoder ϕ on S such that

$$\bar{L}(\phi, p) \leq H(p) + 1.$$

Let $D = (V, E)$ be any β -regular Bratteli diagram, where $\beta \geq 2$. Let $n \geq 0$. If ϕ_n is any lossless prefix encoder on V_n and λ is any PMF on V_n , we define the number

$$R(\phi_n, \lambda) \triangleq \beta^{-n} \bar{L}(\phi_n, \lambda).$$

We discuss the interpretation of $R(\phi_n, \lambda)$. Since D is isomorphic to a canonical diagram, we have

$$|V_n| \leq |V_0|^{\beta^n}, \quad n \geq 0.$$

Thus, there exists a default lossless prefix encoder on V_n employing fixed-length binary codewords of length $\lceil \beta^n \log_2 |V_0| \rceil$. $R(\phi_n, \lambda) / \log_2 |V_0|$ is roughly (for large n) the ratio between the expected codeword length $\bar{L}(\phi_n, \lambda)$ afforded by encoder ϕ_n and the codeword length afforded by the default encoder. It is thus sensible for us to interpret the quantity $R(\phi_n, \lambda)$ as a measure of *compression rate*. A lossless encoding scheme on D is defined to be any sequence $\{\phi_n : n \geq 0\}$ in which ϕ_n is a lossless prefix encoder on V_n ($n \geq 0$). Moreover, if $\{\phi_n\}$ is such a scheme and $\mu \in \mathcal{S}(D)$, by (e.1) we have

$$\liminf_{n \rightarrow \infty} R(\phi_n, \mu_n) \geq H_\infty(\mu). \quad (3)$$

Thus, a scheme $\{\phi_n\}$ which yields the best asymptotic compression rate in encoding the source μ would be one for which

$$\lim_{n \rightarrow \infty} R(\phi_n, \mu_n) = H_\infty(\mu). \quad (4)$$

Such a scheme exists by (e.2). In summary, $H_\infty(\mu)$ has operational significance of being the minimum asymptotic compression rate afforded by lossless encoding schemes used to compress μ .

1.5 Organization of Paper

In the rest of the paper, we prove the six theorems for B-V sources that are stated below. Each of them has an obvious analogue for the stationary sequential sources in $\mathcal{P}(A^\infty)$. The first two theorems are proved in Section II and are as follows.

Theorem 1.5: Weak Universal Encoding Theorem. *Let $D = (V, E)$ be any regular Bratteli diagram. For each $n \geq 0$, there exists a lossless prefix encoder ϕ_n on V_n such that*

$$\lim_{n \rightarrow \infty} R(\phi_n, \mu_n) = H_\infty(\mu), \quad \mu \in \mathcal{S}(D).$$

Theorem 1.6: Strong Universal Encoding Theorem. *Let $D = (V, E)$ be any regular Bratteli diagram. Let Λ be a closed subset of topological space $\mathcal{S}(D)$ such that*

the function $\mu \rightarrow H_\infty(\mu)$ on Λ is a continuous function. For each $n \geq 0$, there exists a lossless prefix encoder ϕ_n on V_n such that

$$\lim_{n \rightarrow \infty} \left\{ \sup_{\mu \in \Lambda} \frac{\bar{L}(\phi_n, \mu_n) - H(\mu_n)}{\beta^n} \right\} = 0.$$

Remarks.

- There are well known weak universal and strong universal encoding theorems for sequential stationary sources [6] that are analogues of these two results. We show in Section II that the sequential source results follow from the above B-V source results.
- In Section II, we show how one may use a simplicial grid to construct a family of B-V sources satisfying the hypotheses of the Strong Universal Lossless Encoding Theorem. The family will be in one-to-one correspondence with the underlying space of the grid.

Section III presents a sufficient amount of the theory of Bratteli-Vershik systems that will allow us to prove the remaining results. Section IV presents the follow two decomposition theorems.

Theorem 1.7: Ergodic Decomposition Theorem. *Let $D = (V, E)$ be any regular Bratteli diagram. For each $\mu \in \mathcal{S}(D)$, there exists a unique probability measure λ_μ on $\mathcal{S}_e(D)$ such that*

$$\mu(v) = \int_{\mathcal{S}_e(D)} \sigma(v) d\lambda_\mu(\sigma), \quad v \in V. \quad (5)$$

Theorem 1.8: Entropy Rate Decomposition Theorem. *Let $D = (V, E)$ be any regular Bratteli diagram. Then*

$$H_\infty(\mu) = \int_{\mathcal{S}_e(D)} H_\infty(\sigma) d\lambda_\mu(\sigma), \quad \mu \in \mathcal{S}(D).$$

There is both a strong and weak form of the Shannon-McMillan-Breiman theorem (SMB theorem) for B-V sources. The strong form gives a limit theorem in which there is both almost everywhere convergence and L^1 convergence. The weak form gives convergence in distribution, and is a consequence of the strong form. Section V establishes the strong form of the SMB theorem for B-V sources. However, the statement of the strong form requires the Section III background material concerning Bratteli-Vershik systems in order for the statement to make sense. Here, we state the weak form and refer the reader to Section V for the strong form.

Theorem 1.9: SMB Theorem (Weak Form). *Let $D = (V, E)$ be a β -regular Bratteli diagram, where $\beta \geq 2$. Let $\mu \in \mathcal{S}(D)$, and let $F_\mu : \mathbb{R} \rightarrow [0, 1]$ be defined by*

$$F_\mu(x) \triangleq \lambda_\mu(\{\sigma \in \mathcal{S}_e(D) : H_\infty(\sigma) \leq x\}), \quad x \in \mathbb{R}. \quad (6)$$

Then

$$\lim_{n \rightarrow \infty} \mu_n(\{v \in V_n : -\beta^{-n} \log_2 \mu_n(v) \leq x\}) = F_\mu(x)$$

for every $x \in \mathbb{R}$ at which F_μ is continuous.

In Section VI, our final section, the following result on fixed-length lossy encoding of B-V sources is established, which requires the SMB theorem for its proof. It is an analogue of a result of Parthasarathy [7] on fixed-length lossy encoding of stationary sequential sources.

Theorem 1.10: Fixed-Length Lossy Encoding Theorem. *Let $D = (V, E)$ be any β -regular Bratteli diagram, where $\beta \geq 2$. Let $\mu \in \mathcal{S}(D)$. For each $\delta \in (0, 1)$, define*

$$M_n(\delta, \mu) \triangleq \min\{|S| : S \subset V_n, \mu_n(S) \geq 1 - \delta\}, \quad n \geq 0. \quad (7)$$

For any $\delta \in (0, 1)$,

$$A(\delta) \leq \liminf_{n \rightarrow \infty} \frac{\log_2 M_n(\delta, \mu)}{\beta^n} \leq \overline{\lim}_{n \rightarrow \infty} \frac{\log_2 M_n(\delta, \mu)}{\beta^n} \leq B(\delta),$$

where

$$\begin{aligned} B(\delta) &\triangleq \inf\{x \in \mathbb{R} : F_\mu(x) > 1 - \delta\}, \\ A(\delta) &\triangleq \sup\{x \in \mathbb{R} : F_\mu(x) < 1 - \delta\}. \end{aligned}$$

Moreover, the set of δ 's for which $A(\delta) \neq B(\delta)$ is countable.

2 Universal Lossless B-V Source Encoding

We consider two types of universal encoding for families of Bratteli-Vershik sources, namely, weak universal encoding and strong universal encoding. As corollaries of our universal encoding results for B-V sources, we will then obtain some previously known universal lossless encoding theorems for stationary sequential source families, thereby illustrating that some parts of universal encoding theory for stationary sequential information sources are special cases of universal encoding theory for Bratteli-Vershik sources.

2.1 Weak Universal Encoding Theory

If ψ is a lossless prefix encoder on finite set S , and τ is a PMF on S , the *redundancy of ψ with respect to τ* is the non-negative real number defined by

$$\text{RED}(\psi, \tau) \triangleq \bar{L}(\psi, \tau) - H(\tau).$$

Let $\beta \geq 2$ and let $D = (V, E)$ be a β -regular Bratteli diagram. A lossless encoding scheme $\{\phi_n\}$ on D is defined to be *weak universal for $\mathcal{S}(D)$* if

$$\lim_{n \rightarrow \infty} \beta^{-n} \text{RED}(\phi_n, \mu_n) = 0, \quad \mu \in \mathcal{S}(D), \quad (8)$$

or equivalently, if

$$\lim_{n \rightarrow \infty} R(\phi_n, \mu_n) = H_\infty(\mu), \quad \mu \in \mathcal{S}(D). \quad (9)$$

The goal of this subsection is to prove the following theorem.

Theorem 2.1. *Let $D = (V, E)$ be any β -regular Bratteli diagram, where $\beta \geq 2$. Then there exists a lossless encoding scheme on D which is weak universal for $\mathcal{S}(D)$.*

For the rest of this section, we fix β -regular Bratteli diagram $D = (V, E)$ and an indexing I of D . For each $x \in V^+$ and $i \in \{0, 1, \dots, \beta - 1\}$, let $e_i(x)$ denote the edge in $E(x)$ such that $I(e_i(x)) = i$ and let $x[i] = s(e_i(x))$. If $n \geq 0$ and ϕ is a lossless prefix encoder on V_n , let $[\phi] : V_{n+1} \rightarrow \{0, 1\}^*$ be the mapping defined by

$$[\phi](x) \triangleq \phi(x[0])\phi(x[1]) \cdots \phi(x[\beta - 1]), \quad x \in V_{n+1}.$$

Since the mapping $x \rightarrow (x[0], x[1], \dots, x[\beta - 1])$ is a one-to-one mapping of V^+ into V^β , $[\phi]$ is a lossless prefix encoder on V_{n+1} .

Lemma 2.2. *Let $n \geq 0$ and let ϕ be a lossless prefix encoder on V_n . Then*

$$R([\phi], \lambda) = R(\phi, [\lambda]) \quad (10)$$

for every PMF λ on V_{n+1} .

Proof. Let X be a V_{n+1} -valued random object whose PMF is λ . We have

$$\beta^{n+1} R([\phi], \lambda) = E \left[\sum_{i=0}^{\beta-1} |\phi(X[i])| \right] = \sum_{i=0}^{\beta-1} E[|\phi(X[i])|].$$

Each $X[i]$ is a V_n -valued random object. The average of the PMF's of the $X[i]$'s is the PMF $[\lambda]$ on V_n , that is,

$$\beta^{-1} \sum_{i=0}^{\beta-1} \Pr[X[i] = u] = [\lambda](u), \quad u \in V_n.$$

Thus, for any real-valued function f on V_n ,

$$\beta^{-1} \sum_{i=0}^{\beta-1} E[f(X[i])] = \sum_{u \in V_n} f(u) [\lambda](u).$$

In the preceding equation, choose f to be the function $u \rightarrow |\phi(u)|$, giving us

$$\beta^{-1} \sum_{i=0}^{\beta-1} E[|\phi(X[i])|] = \beta^n R(\phi, [\lambda]).$$

Equation (10) is now apparent.

For $n \geq 0$, we define $\mathcal{E}(V_n)$ to be the set of all lossless prefix encoders on V_n . An encoder ϕ in $\mathcal{E}(V_n)$ is said to be a *proper encoder* if

$$\sum_{v \in V_n} 2^{-|\phi(v)|} = 1.$$

We define $\mathcal{E}^*(V_n)$ to be the set of all proper encoders in $\mathcal{E}(V_n)$. The concept of proper encoder will be useful to us in the following respect: given any encoder $\phi \in \mathcal{E}(V_n)$, there exists an encoder $\phi^* \in \mathcal{E}^*(V_n)$ such that

$$|\phi^*(v)| \leq |\phi(v)|, \quad v \in V_n.$$

Define

$$\begin{aligned}\mathcal{E}(D) &\triangleq \bigcup_{n=0}^{\infty} \mathcal{E}(V_n) \\ \mathcal{E}^*(D) &\triangleq \bigcup_{n=0}^{\infty} \mathcal{E}^*(V_n)\end{aligned}$$

If $\phi \in \mathcal{E}(D)$, we define its order $\text{ORD}(\phi)$ to be the integer $n \geq 0$ such that $\phi \in \mathcal{E}(V_n)$. Given $\phi \in \mathcal{E}(D)$ and $\mu \in \mathcal{S}(D)$, we define

$$R(\phi, \mu) \triangleq R(\phi, \mu_n),$$

where $n = \text{ORD}(\phi)$. Given $\phi \in \mathcal{E}(D)$, let $\{\phi^{(n)} : n \geq 0\}$ be the sequence in $\mathcal{E}(D)$ defined recursively by

$$\begin{aligned}\phi^{(0)} &= \phi \\ \phi^{(n)} &= [\phi^{(n-1)}], \quad n \geq 1\end{aligned}$$

From Lemma 2.2, we have the properties

$$\text{ORD}(\phi^{(n)}) = \text{ORD}(\phi) + n, \quad \phi \in \mathcal{E}(D), \quad n \geq 0.$$

$$R(\phi^{(n)}, \mu) = R(\phi, \mu), \quad \phi \in \mathcal{E}(D), \quad \mu \in \mathcal{S}(D), \quad n \geq 0.$$

Lemma 2.3. *There is an upper triangular array*

$$\{\phi_{i,j} : j = i, i+1, i+2, \dots; i = 1, 2, 3, \dots\} \subset \mathcal{E}(D)$$

such that

- **(a):** $\text{ORD}(\phi_{i,j}) = j - 1$ for any $\phi_{i,j}$ in the array.
- **(b):** $\phi_{i,j} = \phi_{i,i}^{(j-i)}$ for any $\phi_{i,j}$ in the array.
- **(c):** If $\phi \in \mathcal{E}^*(D)$, then there exists $i \geq 1$ and $k \geq 0$ such that $\phi^{(k)} = \phi_{i,i}$.

Proof. Since each set $\mathcal{E}^*(V_n)$ is finite, we may fix an enumeration $\{\psi_i : i \geq 0\}$ of $\mathcal{E}^*(D)$ such that $\text{ORD}(\psi_i) \leq \text{ORD}(\psi_{i'})$ holds whenever $i < i'$. Then we have $\text{ORD}(\psi_i) \leq i$ for all $i \geq 0$. Define

$$\phi_{i,i} \triangleq \psi_{i-1}^{(i-1-\text{ORD}(\psi_{i-1}))}, \quad i \geq 1.$$

Define $\sigma_{i,j}$ for $j > i$ so that property (b) holds. Then, properties (a) and (c) also hold.

Suppose S is a finite set and $\{\phi_j : 1 \leq j \leq J\}$ is a finite sequence of lossless prefix encoders on S , where $J \geq 2$. For each $j \in \{1, 2, \dots, J\}$, let B_j be the binary string of length $\lceil \log_2 J \rceil$ which is the binary expansion of integer $j - 1$ (most significant bit on the left). For each $s \in S$, let $j(s)$ be the smallest $j \in \{1, \dots, J\}$ such that

$$|\phi_j(s)| = \min\{|\phi_{j'}(s)| : 1 \leq j' \leq J\}.$$

Then we have the lossless prefix encoder ϕ on S defined by

$$\phi(s) \triangleq B(s)\phi_{j(s)}(s), \quad s \in S.$$

This encoder ϕ constructed from $\{\phi_j : 1 \leq j \leq J\}$ shall henceforth be denoted by $\bigwedge_{j=1}^J \phi_j$. We have the property

$$|(\bigwedge_{j=1}^J \phi_j)(s)| \leq \lceil \log_2 J \rceil + \min_{1 \leq j \leq J} |\phi_j(s)|, \quad s \in S. \quad (11)$$

Proof of Theorem 2.1. Let $\{\phi_{i,j}\}$ be any triangular array of encoders in $\mathcal{E}(D)$ chosen according to Lemma 2.3. Let $\{\tau_n : n \geq 0\}$ be the lossless encoding scheme on D defined by

$$\tau_n \triangleq \bigwedge_{i=1}^{n+1} \phi_{i,n+1}, \quad n \geq 0.$$

Let $\mu \in \mathcal{S}(D)$ be arbitrary. We show that

$$\overline{\lim}_{n \rightarrow \infty} R(\tau_n, \mu) \leq H_\infty(\mu), \quad (12)$$

which will imply that $\{\tau_n\}$ is weak universal for $\mathcal{S}(D)$. Let $\varepsilon > 0$ be arbitrary. Pick $N \geq 0$ large enough so that

$$\begin{aligned} \beta^{-N} H(\mu_N) &\leq H_\infty(\mu) + \varepsilon, \\ \beta^{-N} &< \varepsilon \end{aligned}$$

both hold. There exists $\phi \in \mathcal{E}_N^*(D)$ such that

$$R(\phi, \mu) \leq \beta^{-N} (H(\mu_N) + 1) < H_\infty(\mu) + 2\varepsilon.$$

By property (c) of Lemma 2.3, there exists $M \geq 1$ such that $\phi_{M,M} \in \{\phi^{(k)} : k \geq 0\}$. We have

$$R(\phi_{M,M}, \mu) = R(\phi, \mu) < H_\infty(\mu) + 2\varepsilon. \quad (13)$$

Let $m \geq M$. By (11),

$$|\tau_{m-1}| \leq |\phi_{M,m}| + \lceil \log_2 m \rceil,$$

from which it follows that

$$R(\tau_{m-1}, \mu) \leq R(\phi_{M,m}, \mu) + \beta^{-(m-1)} \lceil \log_2 m \rceil.$$

But

$$R(\phi_{M,m}, \mu) = R(\phi_{M,M}, \mu).$$

In view of (13), we have proved that

$$R(\tau_{m-1}, \mu) \leq \beta^{-(m-1)} \lceil \log_2 m \rceil + H_\infty(\mu) + 2\varepsilon, \quad m \geq M,$$

from which it follows that

$$\overline{\lim}_{n \rightarrow \infty} R(\tau_n, \mu) \leq H_\infty(\mu) + 2\varepsilon.$$

Since this statement must be true for every $\varepsilon > 0$, (12) follows, completing the proof that $\{\tau_n\}$ is weak universal for $\mathcal{S}(D)$.

2.2 Strong Universal Encoding Theory

Let $\beta \geq 2$ and let $D = (V, E)$ be a β -regular Bratteli diagram. Let Λ be a subfamily of the Bratteli-Vershik source family $\mathcal{S}(D)$. Let $\{\phi_n : n \geq 0\}$ be a lossless encoding scheme on D . $\{\phi_n\}$ is defined to be *strong universal for Λ* if

$$\lim_{n \rightarrow \infty} \left\{ \sup_{\mu \in \Lambda} \beta^{-n} \text{RED}(\phi_n, \mu_n) \right\} = 0.$$

Theorem 2.4. *Let $D = (V, E)$ be any β -regular Bratteli diagram, where $\beta \geq 2$. Let Λ be a closed subset of $\mathcal{S}(D)$. Let $H_\Lambda : \Lambda \rightarrow [0, \infty)$ be the function*

$$H_\Lambda(\mu) \triangleq H_\infty(\mu), \quad \mu \in \Lambda.$$

If H_Λ is continuous, then there exists a lossless encoding scheme on D which is strong universal for Λ .

Proof. Let $\{\phi_{i,j}\}$ be any triangular array of encoders in $\mathcal{E}(D)$ chosen according to Lemma 2.3. Let $\{\tau_n : n \geq 0\}$ be the lossless encoding scheme on D defined by

$$\tau_n \triangleq \bigwedge_{i=1}^{n+1} \phi_{i,n+1}, \quad n \geq 0.$$

For each $j \geq 1$, let $Q_j : \mathcal{S}(D) \rightarrow [0, \infty)$ be the continuous function

$$Q_j(\mu) \triangleq \min_{1 \leq i \leq j} R(\phi_{i,j}, \mu), \quad \mu \in \mathcal{S}(D).$$

Then we have the monotonicity property

$$Q_1(\mu) \geq Q_2(\mu) \geq Q_3(\mu) \geq \cdots, \quad \mu \in \mathcal{S}(D).$$

Also,

$$\lim_{j \rightarrow \infty} Q_j(\mu) = H_\infty(\mu), \quad \mu \in \mathcal{S}(D).$$

For each $j \geq 1$, let $H_j : \mathcal{S}(D) \rightarrow [0, \infty)$ be the continuous function

$$H_j(\mu) \triangleq \beta^{-(j-1)} H(\mu_{j-1}), \quad \mu \in \mathcal{S}(D).$$

We have

$$H_1(\mu) \geq H_2(\mu) \geq H_3(\mu) \geq \cdots, \quad \mu \in \mathcal{S}(D),$$

and

$$\lim_{j \rightarrow \infty} H_j(\mu) = H_\infty(\mu), \quad \mu \in \mathcal{S}(D).$$

Since the function H_Λ is continuous and Λ is compact, Dini's theorem [26, Thm. 7.2.5] tells us that each of the two monotone sequences of functions $\{Q_j\}$ and $\{H_j\}$ converges

uniformly on Λ to the function H_Λ . Therefore, the sequence $\{Q_j - H_j\}$ converges uniformly to 0 on Λ , that is,

$$\lim_{n \rightarrow \infty} \left[\sup_{\mu \in \Lambda} \{Q_{n+1}(\mu) - H_{n+1}(\mu)\} \right] = 0. \quad (14)$$

Let $\mu \in \Lambda$. The relationships

$$\beta^{-n} \text{RED}(\tau_n, \mu_n) = R(\tau_n, \mu_n) - H_{n+1}(\mu),$$

$$R(\tau_n, \mu_n) \leq \beta^{-n} \lceil \log_2(n+1) \rceil + Q_{n+1}(\mu)$$

hold for $n \geq 0$. Hence,

$$0 \leq \sup_{\mu \in \Lambda} \beta^{-n} \text{RED}(\tau_n, \mu_n) \leq \beta^{-n} \lceil \log_2(n+1) \rceil + \sup_{\mu \in \Lambda} \{Q_{n+1}(\mu) - H_{n+1}(\mu)\}.$$

Since the left side converges to zero as $n \rightarrow \infty$, $\{\tau_n\}$ is strong universal for Λ .

Definition. A family Λ of Bratteli-Vershik sources on a regular Bratteli diagram D is said to be *strongly universally encodable* if there exists a lossless encoding scheme on D which is strong universal for Λ . We present three examples of B-V source families which can be seen to be strongly universally encodable via Theorem 2.4. The third of these examples is of most interest; it shows how a simplicial grid can give rise to a strongly universally encodable family of B-V sources.

Example 2.1. Suppose regular Bratteli diagram D is such that every source in $\mathcal{S}(D)$ has entropy rate 0. Then $\mathcal{S}(D)$ is strongly universally encodable.

Example 2.2. Suppose regular Bratteli diagram D is such that $\mathcal{S}_e(D)$ is finite. Then $\mathcal{S}(D)$ is strongly universally encodable.

Example 2.3. Let β be an integer ≥ 2 . Let $\{K_n : n \geq 0\}$ be a β -admissible simplicial grid in Euclidean space \mathbb{E} . In Section I, for each $n \geq 1$ and $x \in V(K_n)$, we defined the unique multiset $M(x) = \{x_1, x_2, \dots, x_\beta\}$ whose distinct elements are the vertices of a simplex in K_{n-1} and $x = \beta^{-1}(x_1 + \dots + x_\beta)$. Let $S(x)$ be the set of all β -tuples $(u_1, u_2, \dots, u_\beta)$ such that $M(x) = \{u_1, u_2, \dots, u_\beta\}$. For each $n \geq 0$ and $x \in V(K_n)$, we define a subset $C_n(x)$ of $V(K_0)^{\beta^n}$ recursively as follows. The initial sets for the recursion are

$$C_0(x) \triangleq \{x\}, \quad x \in V(K_0).$$

For $n \geq 1$ and $x \in V(K_n)$, we recursively define

$$C_n(x) \triangleq \bigcup_{(u_1, u_2, \dots, u_\beta) \in S(x)} C_{n-1}(u_1) \times C_{n-1}(u_2) \times \dots \times C_{n-1}(u_\beta).$$

Note that if $(z_1, z_2, \dots, z_{\beta^n})$ is any of the β^n -tuples in $C_n(x)$, then $x = \beta^{-n}(z_1 + z_2 + \dots + z_{\beta^n})$. Thus, the collection of sets $\{C_n(x) : x \in V(K_n)\}$ satisfies the property that its members are pairwise disjoint. Let $D = (V, E)$ be the β -regular canonical Bratteli diagram for which

$$V_n = \bigcup_{x \in V(K_n)} C_n(x), \quad n \geq 0.$$

Let Θ be the underlying space of the grid $\{K_n\}$. For each $\theta \in \Theta$, let $\tau^\theta : V \rightarrow [0, 1]$ be the mapping defined by

$$\tau^\theta(u) \triangleq \frac{p_{\theta|K_n}(x)}{|C_n(x)|}, \quad u \in C_n(x), \quad x \in V(K_n), \quad n \geq 0.$$

It can be checked that τ^θ is a source in $\mathcal{S}(D)$. Let Λ be the subfamily of $\mathcal{S}(D)$ defined by

$$\Lambda \triangleq \{\tau^\theta : \theta \in \Theta\}. \quad (15)$$

Let $H_\Theta : \Theta \rightarrow [0, \infty)$ be the function

$$H_\Theta(\theta) \triangleq H_\infty(\tau^\theta), \quad \theta \in \Theta. \quad (16)$$

Λ is a closed subset of topological space $\mathcal{S}(D)$ and the function H_Θ is continuous. Thus, the family of B-V sources Λ is strongly universally encodable by Theorem 2.4. To illustrate, let $\{K_n : n \geq 0\}$ be the $(\beta, k) = (2, 1)$ Kuhn grid, whose underlying space is $\Theta = [0, 1]$. The resulting entropy rate function $H_\Theta : [0, 1] \rightarrow [0, \infty)$ is plotted in Figure 4. The paper [27] explains how to employ iterated function system theory to quickly generate millions of points on the H_Θ curve.

Remarks.

- One can construct a specific strong universal encoding scheme for Λ of (15) by adapting the tree-based compression scheme put forth in [27, Fig. 1]. (Letting $t(n)$ be the finite rooted ordered tree with β^n leaves and β children for each non-leaf, for each $x \in V_n$, we will label the vertices of $t(n)$ in a certain way in our Section V proof of the SMB theorem, starting with label x on the root. The scheme of [27] compresses x by compressing this labeled tree.)
- With some further assumptions on the β -admissible grid $\{K_n\}$, the entropy rate function H_Θ defined in (16) will be a self-affine function (a function whose graph is the attractor of an iterated function system consisting of affine functions [28, Chap. 11]). An infinite collection of self-affine entropy rate functions arising from simplicial grids are catalogued in [27], including the function $H_\Theta : [0, 1] \rightarrow [0, \infty)$ given at the end of Example 2.3.

2.3 Universal Sequential Source Encoding Schemes

Let A be a finite set with at least two elements, and let $D_2(A) = (V, E)$ be the 2-canonical Bratteli diagram $D_2(A)$ of Example 1.3. This is the diagram such that $V_n = A^{2^n}$ ($n \geq 0$), and such that the two edges with range vertex $x \in V_n$ ($n \geq 1$) have source vertices $x[0], x[1]$ in V_{n-1} , where $(x[0], x[1])$ is the 2-decomposition of x . This subsection shows how we may use universal lossless encoding schemes for subfamilies of the B-V source family $\mathcal{S}(D_2(A))$ to construct universal lossless encoding schemes for subfamilies of the sequential source family $\mathcal{P}(A^\infty)$.

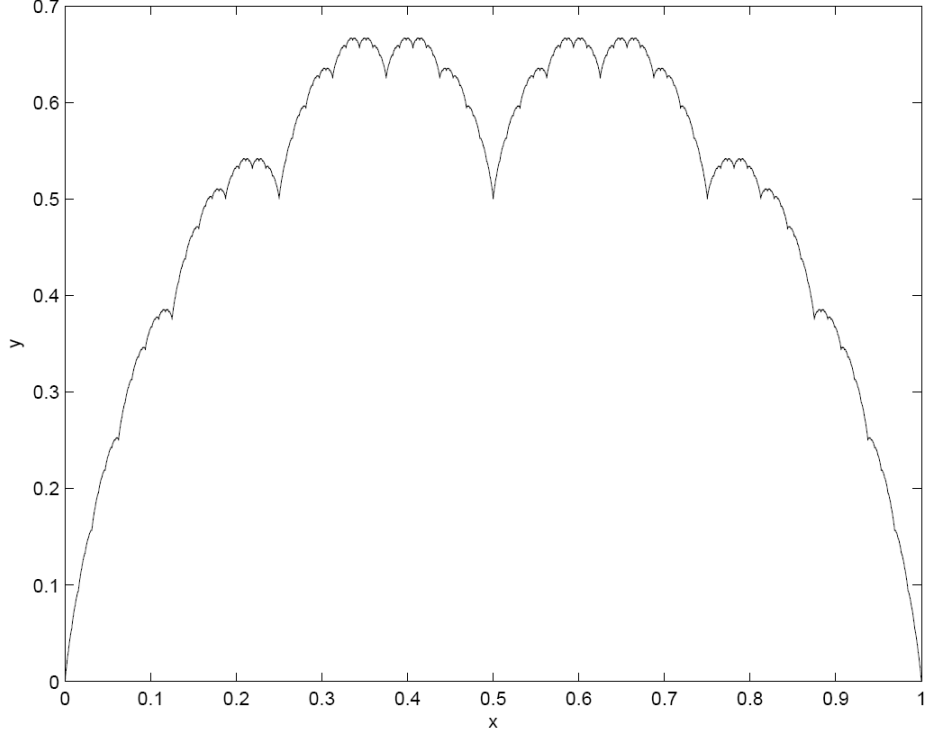


Figure 4: Plot of self affine entropy rate function $H_\Theta : [0, 1] \rightarrow [0, \infty)$ given in Example 2.3.

If α is a source in $\mathcal{P}(A^\infty)$ and $k \geq 1$, then $\alpha^{(k)}$ shall denote the marginal PMF of α on A^k . Furthermore, $H_\infty(\alpha)$ shall denote the entropy rate of α , defined by

$$H_\infty(\alpha) \triangleq \lim_{k \rightarrow \infty} k^{-1} H(\alpha^{(k)}).$$

This limit always exists and is equal to $\inf_k k^{-1} H(\alpha^{(k)})$.

We review universal sequential source encoding concepts put forth in Davisson's ground-breaking paper [6]. A sequence $\{\psi_k : k \geq 0\}$ in which ψ_k is a lossless prefix encoder on A^k ($k \geq 1$) is called a *lossless encoding scheme on A^∞* . A lossless encoding scheme $\{\psi_k\}$ on A^∞ is defined to be *weak universal for $\mathcal{P}(A^\infty)$* [6] if

$$\lim_{k \rightarrow \infty} k^{-1} \text{RED}(\psi_k, \alpha^{(k)}) = 0, \quad \alpha \in \mathcal{P}(A^\infty), \quad (17)$$

or equivalently, if

$$\lim_{k \rightarrow \infty} k^{-1} \bar{L}(\psi_k, \alpha^{(k)}) = H_\infty(\alpha), \quad \alpha \in \mathcal{P}(A^\infty). \quad (18)$$

Let $\mathcal{P}^*(A^\infty)$ be a subfamily of $\mathcal{P}(A^\infty)$. A lossless encoding scheme $\{\psi_k\}$ on A^∞ is

defined to be *strong universal* for $\mathcal{P}^*(A^\infty)$ [6] if

$$\lim_{k \rightarrow \infty} \left\{ \sup_{\alpha \in \mathcal{P}^*(A^\infty)} k^{-1} \text{RED}(\psi_k, \alpha^{(k)}) \right\} = 0.$$

As discussed in Ex. 1.3, each sequential source $\alpha \in \mathcal{P}(A^\infty)$ gives rise to the source $\alpha^\dagger \in \mathcal{S}(D_2(A))$ whose restriction to V_n is $\alpha^{(2^n)}$ ($n \geq 0$). The sources α and α^\dagger have the same entropy rate. The family of sequential sources $\mathcal{P}(A^\infty)$ is thus in one-to-one correspondence with the family of Bratteli-Vershik sources

$$\mathcal{P}(A^\infty)^\dagger = \{\alpha^\dagger : \alpha \in \mathcal{P}(A^\infty)\} \subset \mathcal{S}(D_2(A)).$$

Construction of Schemes on A^∞ from schemes on $D_2(A)$. Let $\{\phi_n : n \geq 0\}$ be any lossless encoding scheme on $D_2(A)$. Using $\{\phi_n\}$, we show how to construct a lossless encoding scheme $\{\psi_k : k \geq 0\}$ on A^∞ . For $k \geq 1$, let S_k be the set of positive integers

$$S_k = \{m(k, 1), m(k, 2), \dots, m(k, J_k)\}$$

whose elements left to right are the decreasing powers of two which sum to k . Factor each $x \in A^k$ as

$$x = x(1)x(2) \cdots x(J_k),$$

where $x(j)$ is of length $m(k, j)$ ($j = 1, 2, \dots, J_k$). Define

$$n(k, j) \triangleq \log_2 m(k, j).$$

For each $k \geq 1$, define the mapping $\psi_k : A^k \rightarrow \{0, 1\}^*$ by

$$\psi_k(x) = \phi_{n(k, 1)}(x(1))\phi_{n(k, 2)}(x(2)) \cdots \phi_{n(k, J_k)}(x(J_k)), \quad x \in A^k. \quad (19)$$

It is straightforward to check that ψ_k is a lossless prefix encoder. Therefore, $\{\psi_k : k \geq 1\}$ is a lossless encoding scheme on A^∞ . If the scheme $\{\phi_n : n \geq 0\}$ on $D_2(A)$ and the scheme $\{\psi_k : k \geq 1\}$ on A^∞ are related in this way, we henceforth state that $\{\phi_n\}$ *induces* $\{\psi_k\}$ or that $\{\psi_k\}$ *is induced by* $\{\phi_n\}$.

The following two results, which are consequences of Theorems 2.1 and 2.4, give circumstances under which a universal scheme on $D_2(A)$ induces a universal scheme on A^∞ . They imply results of Davisson [6, Thm. 7] on the existence of weak and strong universal encoding schemes for sequential source families. This shows that parts of universal encoding theory for sequential sources are special cases of universal encoding theory for Bratteli-Vershik sources.

Theorem 2.5. *Let $\{\phi_n : n \geq 0\}$ be any lossless encoding scheme on $D_2(A)$ which is weak universal for $\mathcal{S}(D_2(A))$. Then the lossless encoding scheme on A^∞ induced by $\{\phi_n\}$ is weak universal for $\mathcal{P}(A^\infty)$.*

Theorem 2.6. *Let Γ be a subfamily of $\mathcal{P}(A^\infty)$ closed with respect to the weak topology on $\mathcal{P}(A^\infty)$. Let $H_\Gamma : \Lambda \rightarrow [0, \infty)$ be the function*

$$H_\Gamma(\alpha) \triangleq H_\infty(\alpha), \quad \alpha \in \Gamma.$$

Suppose that H_Γ is a continuous function. Then there exists a lossless encoding scheme on $D_2(A)$ which is strong universal for Γ^\dagger . Furthermore, any such scheme induces a lossless encoding scheme on A^∞ which is strong universal for Γ .

Proof of Theorem 2.5. Let $\{\phi_n : n \geq 0\}$ be a scheme on $\mathcal{S}(D_2(A))$ which is weak universal for $\mathcal{S}(D_2(A))$. Let $\{\psi_k : k \geq 1\}$ be the scheme on A^∞ induced by $\{\phi_n\}$. Let $\alpha \in \mathcal{P}(A^\infty)$ be arbitrary and let $\varepsilon > 0$ be arbitrary. The proof is completed by showing that

$$\overline{\lim}_{k \rightarrow \infty} k^{-1} \bar{L}(\psi_k, \alpha^{(k)}) \leq H_\infty(\alpha) + \varepsilon. \quad (20)$$

Since the scheme $\{\phi_n\}$ is weak universal for $\mathcal{S}(D_2(A))$,

$$\lim_{n \rightarrow \infty} R(\phi_n, \alpha^\dagger) = H_\infty(\alpha^\dagger) = H_\infty(\alpha). \quad (21)$$

As a consequence, there is a positive real number C and a positive integer N such that

$$R(\phi_n, \alpha^\dagger) \leq C, \quad n \geq 0,$$

$$R(\phi_n, \alpha^\dagger) \leq H_\infty(\alpha) + \varepsilon, \quad n \geq N.$$

By (19),

$$k^{-1} \bar{L}(\psi_k, \alpha^{(k)}) = k^{-1} \sum_{m \in S_k} m R(\phi_{\log_2 m}, \alpha^\dagger), \quad k \geq 1. \quad (22)$$

For each $k \geq 1$, we partition S_k into the two sets

$$S_k(1) = \{m \in S_k : m \leq 2^N\}$$

$$S_k(2) = \{m \in S_k : m > 2^N\}$$

We have the bounds

$$\sum_{m \in S_k(1)} m \leq 2^{N+1},$$

$$R(\phi_{\log_2 m}, \alpha^\dagger) \leq H_\infty(\alpha) + \varepsilon, \quad m \in S_k(2).$$

Applying these bounds to the right side of equation (22),

$$k^{-1} \bar{L}(\psi_k, \alpha^{(k)}) \leq k^{-1} 2^{N+1} C + H_\infty(\alpha) + \varepsilon, \quad k \geq 1,$$

from which (20) follows by letting $k \rightarrow \infty$.

Proof of Theorem 2.6. Let $\Lambda = \Gamma^\dagger$. Then Λ is a subfamily of $\mathcal{S}(D_2(A))$ which satisfies the assumptions of Theorem 2.4. Thus, there exists a lossless encoding scheme on $D_2(A)$ which is strong universal for Γ^\dagger . Let $\{\phi_n : n \geq 0\}$ be any such scheme and let $\{\psi_k : k \geq 0\}$ be the lossless encoding scheme on A^∞ induced by $\{\phi_n\}$. Let $\varepsilon > 0$ be arbitrary. The proof is completed by showing that

$$\overline{\lim}_{k \rightarrow \infty} \left\{ \sup_{\alpha \in \Gamma} k^{-1} \text{RED}(\psi_k, \alpha^{(k)}) \right\} \leq \varepsilon. \quad (23)$$

We have

$$k^{-1} \text{RED}(\psi_k, \alpha^{(k)}) \leq k^{-1} \bar{L}(\psi_k, \alpha^{(k)}) - H_\infty(\alpha), \quad \alpha \in \Gamma. \quad (24)$$

By the Dini theorem argument used in the proof of Theorem 2.4, there is a positive integer N_1 such that

$$2^{-n}H(\alpha^{(2^n)}) \leq H_\infty(\alpha) + \varepsilon/2, \quad n \geq N_1, \quad \alpha \in \Gamma.$$

Since $\{\phi_n\}$ is strong universal for Γ^\dagger , there is a positive integer N_2 such that

$$R(\phi_n, \alpha^\dagger) \leq 2^{-n}H(\alpha^{(2^n)}) + \varepsilon/2, \quad n \geq N_2, \quad \alpha \in \Gamma.$$

Let $N = \max(N_1, N_2)$. Then we have

$$R(\phi_n, \alpha^\dagger) \leq H_\infty(\alpha) + \varepsilon, \quad n \geq N, \quad \alpha \in \Gamma.$$

The mapping $\alpha \rightarrow \max_{0 \leq n < N} R(\phi_n, \alpha)$ is a continuous mapping on Γ and Γ is compact; therefore this mapping is bounded on Γ . There thus exists a positive real number C such that

$$R(\phi_n, \alpha^\dagger) \leq C, \quad n \geq 0, \quad \alpha \in \Gamma.$$

As argued in the proof of Theorem 2.5,

$$k^{-1}\bar{L}(\psi_k, \alpha^{(k)}) \leq k^{-1}2^{N+1}C + H_\infty(\alpha) + \varepsilon, \quad k \geq 1, \quad \alpha \in \Gamma.$$

Applying inequality (24), it follows that

$$\sup_{\alpha \in \Gamma} k^{-1}\text{RED}(\psi_k, \alpha^{(k)}) \leq k^{-1}2^{N+1}C + \varepsilon, \quad k \geq 1.$$

Letting $k \rightarrow \infty$, (23) holds and our proof is complete.

3 Some Bratteli-Vershik System Theory

In this section, we explain how a regular Bratteli diagram D induces the Bratteli-Vershik dynamical systems that were alluded to in Section I. We develop some theory for these systems, including the fact that there is a one-to-one correspondence between the sources in $\mathcal{S}(D)$ and the Bratteli-Vershik systems induced by D . In later sections, we will be able to establish results about a Bratteli-Vershik source μ by exploiting dynamical properties of the Bratteli-Vershik system corresponding to μ .

Let β be an integer ≥ 2 and let S_β be the set $\{0, 1, \dots, \beta - 1\}$. Throughout this section, $D = (V, E)$ is a fixed β -regular Bratteli diagram and $I : E \rightarrow S_\beta$ is any fixed embedding. For each vertex $x \in V^+$ and $i \in S_\beta$, $e_i(x)$ denotes the unique edge in $E(x)$ such that $I(e_i(x)) = i$ and $x[i]$ denotes the vertex $s(e_i(x))$. The β -tuple $(x[0], x[1], \dots, x[\beta - 1])$ is called the β -decomposition of x ; we have $M_D(x) = \{x[0], \dots, x[\beta - 1]\}$.

In the following, if (U_0, U_1, \dots, U_n) is a deterministic or random sequence of finite length and $0 \leq i \leq j \leq n$, then $U_i^j \triangleq (U_i, \dots, U_j)$. If (U_0, U_1, U_2, \dots) is an infinite sequence and $i \geq 0$, then $U_i^\infty \triangleq (U_i, U_{i+1}, \dots)$ and $U_0^i = (U_0, \dots, U_i)$.

3.1 β -Expansions and β -adic Arithmetic

Let $n \geq 1$. Define

$$S_{\beta,n} \triangleq \{0, 1, \dots, \beta^n - 1\}.$$

The set of integers $S_{\beta,n}$ and the set of n -tuples S_{β}^n are both of cardinality β^n . Each integer $i \in S_{\beta,n}$ has a unique expansion

$$[i]_{\beta,n} = (i_0, i_1, \dots, i_{n-1}) \in S_{\beta}^n$$

into digits from S_{β} in which

$$i = i_0 + i_1\beta + \dots + i_{n-1}\beta^{n-1}. \quad (25)$$

We call $[i]_{\beta,n}$ the β -expansion of i . Note that in going from left to right in the expansion (i_0, \dots, i_{n-1}) , we are going from least significant digit i_0 to most significant digit i_{n-1} . (Thus, for example, we write $[35]_{2,4} = 110001$ instead of reversed as 100011.) The mapping $i \rightarrow [i]_{\beta,n}$ is a one-to-one mapping of set $S_{\beta,n}$ onto set S_{β}^n .

Listing the n -tuples in S_{β}^n in lexicographical order and then reversing the order of the entries in each n -tuple, the list

$$L(n) \triangleq ([0]_{\beta,n}, [1]_{\beta,n}, \dots, [\beta^n - 1]_{\beta,n})$$

is obtained. For example, if $\beta = 3$ and $n = 2$, the lexicographical ordering of S_{β}^n is

$$00, 01, 02, 10, 11, 12, 20, 21, 22.$$

Reversing these, we obtain the list

$$L(2) = (00, 10, 20, 01, 11, 21, 02, 12, 22)$$

giving the expansions of $0, 1, \dots, 8$, respectively.

Suppose that $i, i+1$ belong to $S_{\beta,n}$ and we are given the expansion $[i]_{\beta,n}$. It is desired to compute the expansion $[i+1]_{\beta,n}$ directly from the expansion $[i]_{\beta,n}$. In the rest of this subsection, we develop an efficient method for accomplishing this computation which exploits β -adic arithmetic.

Let $(\mathbb{Z}_{\beta}, \oplus)$ be the additive group of the β -adic integers [29, Chap. 1]. \mathbb{Z}_{β} consists of all unilateral infinite sequences $\mathbf{z} = (z_0, z_1, z_2, \dots)$ in which all $z_i \in S_{\beta}$. The entries z_i are called the β -adic digits of \mathbf{z} . The β -adic addition operation \oplus on \mathbb{Z}_{β} operates similarly to the usual addition algorithm for integers expanded into decimal digits, except our convention is to carry β -adic digits from left to right rather than right to left. For example, if $\beta = 3$,

$$(1, 0, 1, 2, 1, \dots) \oplus (2, 1, 0, 1, 1, \dots) = (0, 2, 1, 0, 0, \dots).$$

If $\beta = 7$, we have

$$(3, 3, 2, 5, 1, \dots) \oplus (6, 2, 0, 2, 3, \dots) = (2, 6, 2, 0, 5, \dots).$$

Let $\mathbf{1} \in \mathbb{Z}_\beta$ be the β -adic integer (z_0, z_1, z_2, \dots) in which $z_0 = 1$ and all other entries are 0. (There is also a β -adic multiplication operation \otimes on \mathbb{Z}_β such that $(\mathbb{Z}_\beta, \oplus, \otimes)$ is a ring, and $\mathbf{1}$ is the identity element with respect to \otimes .) The *adding machine transformation* on \mathbb{Z}_β [30, Chap. 7] is the one-to-one mapping of \mathbb{Z}_β onto itself in which $\mathbf{z} \in \mathbb{Z}_\beta$ is mapped into $\mathbf{1} \oplus \mathbf{z}$. If $\mathbf{z} = (z_i : i \geq 0)$ contains at least one entry $z_i < \beta - 1$, it is easy to compute $\mathbf{1} \oplus \mathbf{z}$ via the following three-step procedure: (i) find the first entry z_i of \mathbf{z} which is $< \beta - 1$; (ii) set all entries preceding z_i equal to zero; (iii) increase z_i by 1 and keep all subsequent entries unchanged. To illustrate, for $\beta = 2$, we have

$$\mathbf{1} \oplus (0, 1, 0, 1, 1, z_5, z_6, \dots) = (1, 1, 0, 1, 1, z_5, z_6, \dots),$$

$$\mathbf{1} \oplus (1, 1, 0, 0, 1, z_5, z_6, \dots) = (0, 0, 1, 0, 1, z_5, z_6, \dots).$$

Let $n \geq 1$. If $\alpha \in S_\beta^n$, let $\alpha 0^\infty$ be the element of \mathbb{Z}_β starting with the entries of α , followed by infinitely many zeroes. The mapping $\alpha \rightarrow \alpha 0^\infty$ is an embedding of S_β^n into \mathbb{Z}_β . Let \tilde{S}_β^n be the set consisting of the $\beta^n - 1$ n -tuples in S_β^n which are not equal to $(\beta - 1, \beta - 1, \dots, \beta - 1)$. If $\alpha \in \tilde{S}_\beta^n$, we define $\mathbf{1} \oplus_n \alpha$ to be the element $\alpha' \in S_\beta^n$ such that

$$\mathbf{1} \oplus \alpha 0^\infty = \alpha' 0^\infty.$$

We have

$$\mathbf{1} \oplus_n [i]_{\beta, n} = [i + 1]_{\beta, n}, \quad 0 \leq i < \beta^n.$$

This formula gives us the efficient method alluded to previously for directly computing $[i + 1]_{\beta, n}$ from $[i]_{\beta, n}$. The list $L(n)$ of all n -tuples in S_β^n referred to earlier is then

$$L(n) = (\alpha_0, \alpha_1, \dots, \alpha_{\beta^n - 1}),$$

where the entries of the list are generated via the recursion

$$\alpha_0 = (0, 0, \dots, 0),$$

$$\alpha_{i+1} = \mathbf{1} \oplus_n \alpha_i, \quad 0 \leq i < \beta^n - 1.$$

For example, let $\beta = 3$ and $n = 3$. Since $\beta^n = 3^3 = 27$, list $L(3)$ takes the form

$$L(3) = (\alpha_0, \alpha_1, \dots, \alpha_{26}).$$

The first ten 3-tuples in the list $L(3)$ are generated as follows:

$$\alpha_0 = (000) = [0]_{3,3},$$

$$\mathbf{1} \oplus_3 (000) = (100) = \alpha_1 = [1]_{3,3}$$

$$\mathbf{1} \oplus_3 (100) = (200) = \alpha_2 = [2]_{3,3}$$

$$\mathbf{1} \oplus_3 (200) = (010) = \alpha_3 = [3]_{3,3}$$

$$\mathbf{1} \oplus_3 (010) = (110) = \alpha_4 = [4]_{3,3}$$

$$\mathbf{1} \oplus_3 (110) = (210) = \alpha_5 = [5]_{3,3}$$

$$\mathbf{1} \oplus_3 (210) = (020) = \alpha_6 = [6]_{3,3}$$

$$\mathbf{1} \oplus_3 (020) = (120) = \alpha_7 = [7]_{3,3}$$

$$\mathbf{1} \oplus_3 (120) = (220) = \alpha_8 = [8]_{3,3}$$

$$\mathbf{1} \oplus_3 (220) = (001) = \alpha_9 = [9]_{3,3}$$

The remaining 17 members of the list $L(3)$ are similarly generated.

3.2 Dynamics of Finite Paths in D

A finite path in $D = (V, E)$ is a sequence of finitely many connected edges from E . We shall be interested in finite paths in D which start at level 0 (that is, at some vertex in V_0) and then ascend level-by-level, successively visiting vertices in V_1, V_2, \dots until the terminating vertex is reached. If such a path consists of n edges, then the terminating vertex of the path will lie in vertex set V_n .

Let $n \geq 1$. $\Pi_D(0, n)$ is the set of all paths in D that start at a vertex in V_0 and terminate at a vertex in V_n . That is, the paths in $\Pi_D(0, n)$ are the n -tuples $(\omega_0, \omega_1, \dots, \omega_{n-1})$ such that $\omega_i \in E_i$ ($0 \leq i \leq n-1$) and $r(\omega_i) = s(\omega_{i+1})$ ($0 \leq i \leq n-2$). The *address* $\alpha(y)$ of path $y = (\omega_0, \omega_1, \dots, \omega_{n-1}) \in \Pi_D(0, n)$ is the n -tuple in S_β^n defined by

$$\alpha(y) \triangleq (I(\omega_0), I(\omega_1), \dots, I(\omega_{n-1})).$$

The *index* $i(y)$ of this path y is defined to be the integer computed from the address $\alpha(y)$ via

$$i(y) \triangleq I(\omega_0) + \beta I(\omega_1) + \dots + \beta^{n-1} I(\omega_{n-1}).$$

The index $i(y)$ belongs to the set $S_{\beta, n}$. We have

$$\alpha(y) = [i(y)]_{\beta, n},$$

that is, the address of path y is the β -expansion of the index of y . We can thus compute the index of a path in $\Pi_D(0, n)$ from its address and vice-versa.

Let $n \geq 1$ and $x \in V_n$. $\Pi_D(0, n, x)$ is the set of all paths in D that start at a vertex in V_0 and end at vertex x . That is, $\Pi_D(0, n, x)$ consists of all paths $(\omega_0, \dots, \omega_{n-1})$ in $\Pi_D(0, n)$ such that $r(\omega_{n-1}) = x$. Each path in $\Pi_D(0, n, x)$ is uniquely determined by its address as follows: If path $(\omega_0, \dots, \omega_{n-1}) \in \Pi_D(0, n, x)$ has address (i_0, \dots, i_{n-1}) , the path is computed via the backward recursion

$$\omega_{n-1} = e_{i_{n-1}}(x), \quad (26)$$

$$\omega_j = e_{i_j}(s(\omega_{j+1})), \quad 0 \leq j \leq n-2. \quad (27)$$

Alternatively, each path in $\Pi_D(0, n, x)$ is uniquely determined by its index as follows: If path $y \in \Pi_D(0, n, x)$ has index i , the address of y is then $[i]_{\beta, n}$, from which y is determined by the above backward recursion. The mapping which maps each path in $\Pi_D(0, n, x)$ into its address is a one-to-one mapping of $\Pi_D(0, n, x)$ onto S_β^n . For each $\alpha \in S_\beta^n$, we let $y[\alpha, x]$ denote the path in $\Pi_D(0, n, x)$ whose address is α . For each $i \in S_{\beta, n}$, we let $y(i, x)$ denote the path in $\Pi_D(0, n, x)$ whose index is i .

The path sets $\{\Pi_D(0, n, x) : x \in V_n\}$ are pairwise disjoint and their union is $\Pi_D(0, n)$. Since each path set $\Pi_D(0, n, x)$ has cardinality β^n , the cardinality of $\Pi_D(0, n)$ is $\beta^n |V_n|$.

Let $n \geq 1$. A path in $\Pi_D(0, n)$ whose address is $(\beta-1, \beta-1, \dots, \beta-1)$ is called a *final path*; equivalently, a path is final if its index is $\beta^n - 1$. A path in $\Pi_D(0, n)$ whose address is $(0, 0, \dots, 0)$ is called an *initial path*; equivalently, a path is initial if its index is 0. Define $\tilde{\Pi}_D(0, n)$ to be the set of paths in $\Pi_D(0, n)$ which are not final. That is,

$$\tilde{\Pi}_D(0, n) \triangleq \{y[\alpha, x] : \alpha \in \tilde{S}_\beta^n, x \in V_n\}.$$

Let $T_n : \tilde{\Pi}_D(0, n) \rightarrow \Pi_D(0, n)$ be the one-to-one mapping defined by

$$T_n(y(i, x)) \stackrel{\Delta}{=} y(i+1, x), \quad 0 \leq i < \beta^n - 1, \quad x \in V_n.$$

Equivalently,

$$T_n(y[\alpha, x]) = y[\mathbf{1} \oplus_n \alpha, x], \quad \alpha \in \tilde{S}_\beta^n, \quad x \in V_n.$$

Thus, if $y \in \tilde{\Pi}_D(0, n, x)$ for some $x \in V_n$, then $T_n(y)$ is the path in $\Pi_D(0, n, x)$ whose address is $\mathbf{1} \oplus_n \alpha(y)$ (which is also the path whose index is $i(y) + 1$). Consequently, given $x \in V_n$, we may start with the initial path in $\Pi_D(0, n, x)$ and iteratively apply the function T_n to dynamically generate the entire set of paths $\{y(i, x) : 0 \leq i \leq \beta^n - 1\}$ comprising $\Pi_D(0, n, x)$.

In the next subsection, we see how to splice together the mappings T_1, T_2, T_3, \dots to obtain a unique transformation mapping a set of infinite paths in D into itself; this transformation is the Vershik transformation.

3.3 Vershik Transformation on Aperiodic Infinite Paths in D

We now consider infinite paths in D that start at vertices in V_0 ; these paths ascend level-by-level, visiting vertices at every level V_n ($n \geq 1$) along the way. Certain of these infinite paths are called aperiodic paths, and it is these aperiodic paths that will be our focus in this subsection.

$\Pi_D(0, \infty)$ denotes the path space consisting of all infinite paths in D which start at a vertex in V_0 . $\Pi_D(0, \infty)$ thus consists of all infinite sequences $\omega = (\omega_0, \omega_1, \omega_2, \dots)$ such that $\omega_0^{n-1} \in \Pi_D(0, n)$ for every $n \geq 1$. The Cartesian product space

$$E(0, \infty) \stackrel{\Delta}{=} E_0 \times E_1 \times E_2 \times \dots$$

is a compact space under the Cartesian product topology, and $\Pi_D(0, \infty)$ is a closed subset of $E(0, \infty)$. Therefore, $\Pi_D(0, \infty)$ is a compact topological space under the topology it inherits from $E(0, \infty)$. The address $\alpha(\omega)$ of $\omega \in \Pi_D(0, \infty)$ is defined to be the β -adic integer $(I(\omega_n) : n \geq 0) \in \mathbb{Z}_\beta$. Equivalently, β -adic integer $\mathbf{z} = (z_n : n \geq 0)$ is the address of $\omega \in \Pi_D(0, \infty)$ if and only if the path $\omega_0^{n-1} \in \Pi_D(0, n)$ has address z_0^{n-1} for every $n \geq 1$. We thus have

$$\begin{aligned} \{\omega \in \Pi_D(0, \infty) : \alpha(\omega) = \mathbf{z}\} = \\ \bigcap_{n=1}^{\infty} \{\omega \in \Pi_D(0, \infty) : \alpha(\omega_0^{n-1}) = z_0^{n-1}\}. \end{aligned}$$

The sets on the right in the preceding equation are non-increasing and non-empty subsets of $\Pi_D(0, \infty)$; they are also compact sets because they are closed subsets of $\Pi_D(0, \infty)$. Thus, the set of paths in $\Pi_D(0, \infty)$ with address \mathbf{z} is a non-empty compact set for every $\mathbf{z} \in \mathbb{Z}_\beta$.

A sequence $\mathbf{z} = (z_n : n \geq 0) \in \mathbb{Z}_\beta$ is defined to be *aperiodic* if it is not eventually periodic, that is, there exists no $N \geq 0$ such that the sequence $(z_n : n \geq N)$ is periodic. A path in $\Pi_D(0, \infty)$ is defined to be an *aperiodic path* if its address is aperiodic. Ω_D is defined to be the set of all aperiodic paths in $\Pi_D(0, \infty)$. Ω_D is uncountable because

$\alpha(\Omega_D)$ is the uncountable set of aperiodic sequences in \mathbb{Z}_β . Ω_D is a topological space under the topology it inherits from $\Pi_D(0, \infty)$.

For $n \geq 1$, $i \in S_{\beta,n}$, $\alpha \in S_\beta^n$, and $x \in V_n$, we define cylinder sets

$$\begin{aligned} C_n(i) &\triangleq \{\omega \in \Omega_D : \alpha(\omega_0^{n-1}) = [i]_{\beta,n}\} \\ C_n(i, x) &\triangleq \{\omega \in \Omega_D : \omega_0^{n-1} = y(i, x)\} \\ C_n[\alpha, x] &\triangleq \{\omega \in \Omega_D : \omega_0^{n-1} = y[\alpha, x]\} \end{aligned}$$

Each of these cylinder sets is a clopen subset of Ω_D (that is, it is both closed and open). The countable collection

$$\{C_n(i, x) : i \in S_{\beta,n}, x \in V_n, n \geq 1\}$$

of cylinder sets is a basis for the topology of Ω_D . We let $\mathcal{F}(\Omega_D)$ denote the sigma-field of subsets of Ω_D generated by this collection of cylinder sets. This gives us the measurable space $(\Omega_D, \mathcal{F}(\Omega_D))$.

If $\omega = (\omega_n : n \geq 0)$ is a path in Ω_D , we define the positive integer $N(\omega)$ by

$$N(\omega) \triangleq \min\{n \geq 1 : I(\omega_{n-1}) < \beta - 1\}.$$

$N(\omega)$ has the following meaning: If n is a positive integer, the path $\omega_0^{n-1} \in \Pi_D(0, n)$ is not a final path if and only if $n \geq N(\omega)$. Thus, $T_n(\omega_0^{n-1})$ is defined as a path in $\Pi_D(0, n)$ if and only if $n \geq N(\omega)$.

Lemma 3.1. $D = (V, E)$ is a β -regular Bratteli diagram. Let $\omega \in \Omega_D$. For $n \geq N(\omega)$,

$$T_{n+1}(\omega_0^n) = (T_n(\omega_0^{n-1}), \omega_n),$$

the path in $\Pi_D(0, n+1)$ obtained by appending edge $\omega_n \in E_n$ to the end of path $T_n(\omega_0^{n-1}) \in \Pi_D(0, n)$.

Proof. Let $\omega \in \Omega_D$, let $n \geq N(\omega)$, and let

$$T_{n+1}(\omega_0^n) = \tilde{\omega} = (\tilde{\omega}_0, \dots, \tilde{\omega}_n),$$

$$T_n(\omega_0^{n-1}) = \hat{\omega} = (\hat{\omega}_0, \dots, \hat{\omega}_{n-1}).$$

We have to show that $\tilde{\omega}_n = \omega_n$ and $\tilde{\omega}_0^{n-1} = \hat{\omega}$. We have

$$\alpha(\tilde{\omega}) = \mathbf{1} \oplus_{n+1} \alpha(\omega_0^n), \tag{28}$$

$$\alpha(\hat{\omega}) = \mathbf{1} \oplus_n \alpha(\omega_0^{n-1}).$$

Thus, $\alpha(\hat{\omega})$ must be a prefix of $\alpha(\tilde{\omega})$ since $\alpha(\omega_0^{n-1})$ is a prefix of $\alpha(\omega_0^n)$. That is, letting

$$\alpha(\tilde{\omega}) = \tilde{\alpha} = (\tilde{\alpha}_0, \dots, \tilde{\alpha}_n)$$

$$\alpha(\hat{\omega}) = \hat{\alpha} = (\hat{\alpha}_0, \dots, \hat{\alpha}_{n-1}),$$

we must have

$$\hat{\alpha} = \tilde{\alpha}_0^{n-1}. \tag{29}$$

Equation (29) tells us that the paths $\hat{\omega}$ and $\tilde{\omega}_0^{n-1}$ have the same address $\hat{\alpha}$. Thus, the two paths $\hat{\omega}$ and $\tilde{\omega}_0^{n-1}$ will be identical if we can show that they end at the same vertex in V_n . Let $x \in V_{n+1}$ be the vertex at which path $\tilde{\omega}$ ends. Since $T_{n+1}(\omega_0^n) = \tilde{\omega}$, path ω_0^n also ends at x , and we thus have

$$r(\omega_n) = r(\tilde{\omega}_n) = x. \quad (30)$$

Let

$$\alpha(\omega_0^n) = (\alpha_0, \dots, \alpha_n).$$

The address of path $\tilde{\omega}$ is computed from the address of path ω_0^n via the computation on the right side of equation (28). This computation can only change at most the first n coordinates of the address of ω_0^n since one of these coordinates is $< \beta - 1$. Thus, $\alpha_n = \tilde{\alpha}_n$, and we have

$$I(\omega_n) = \alpha_n = \tilde{\alpha}_n = I(\tilde{\omega}_n).$$

But by (30), both edges ω_n and $\tilde{\omega}_n$ belong to $E(x)$, so we must have $\omega_n = \tilde{\omega}_n$. Since $\hat{\omega} = T_n(\omega_0^{n-1})$, paths $\hat{\omega}$ and ω_0^{n-1} end at the same vertex in V_n . Since $\omega_n = \tilde{\omega}_n$, paths ω_0^{n-1} and $\tilde{\omega}_0^{n-1}$ end at the same vertex in V_n . Thus, paths $\hat{\omega}$ and $\tilde{\omega}_0^{n-1}$ end at the same vertex in V_n , allowing us to conclude that $\hat{\omega} = \tilde{\omega}_0^{n-1}$ as pointed out earlier.

Definition. The *Vershik transformation* is the mapping $T : \Omega_D \rightarrow \Omega_D$ defined as follows. Let $\omega \in \Omega_D$ and let $N = N(\omega)$. Then

$$T(\omega) \triangleq (T_N(\omega_0^{N-1}), \omega_N^\infty).$$

By Lemma 3.1, we have the property

$$T(\omega) = (T_n(\omega_0^{n-1}), \omega_n^\infty), \quad n \geq N(\omega), \quad \omega \in \Omega_D.$$

Thus, for n sufficiently large, the path formed by the first n components of $T(\omega)$ is $T_n(\omega_0^{n-1})$, and the remaining components of $T(\omega)$ coincide with the corresponding components of ω . That is,

$$T(\omega)_0^{n-1} = T_n(\omega_0^{n-1}), \quad n \geq N(\omega), \quad \omega \in \Omega_D.$$

$$T(\omega)_n^\infty = \omega_n^\infty, \quad n \geq N(\omega), \quad \omega \in \Omega_D.$$

From this, it is clear that the definition of the Vershik transformation that was used represents a reasonable way to splice together the collection of finite path maps $\{T_n : n \geq 1\}$ to obtain a mapping on infinite paths.

T is a one-to-one mapping of Ω_D onto itself and thus the inverse transformation T^{-1} exists. Here is a simple argument to establish this. Let us write $T(I)$ for T to denote its dependence upon the embedding I . Let $J : E \rightarrow S_\beta$ be the embedding $J = \beta - 1 - I$. Then we have another Vershik transformation $T(J) : \Omega_D \rightarrow \Omega_D$. It is straightforward to argue that both $T(I) \circ T(J)$ and $T(J) \circ T(I)$ are the identity transformation on Ω_D . Thus, $T(J) = T^{-1}$.

Properties of the Vershik Transformation. We list some easily established properties of T .

- **(a):** Both T and T^{-1} are continuous, and thus T is a homeomorphism of the topological space Ω_D .
- **(b):** T is bimeasurable (that is, both T and T^{-1} are measurable mappings), and thus T is an automorphism of the measurable space $(\Omega_D, \mathcal{F}(\Omega_D))$.
- **(c):** Let \sim be the equivalence relation on Ω_D such that $\omega \sim \hat{\omega}$ if and only if there exists $n \geq 0$ such that $\omega_n^\infty = \hat{\omega}_n^\infty$. Then

$$\omega \sim T(\omega), \quad \omega \in \Omega_D,$$

and T preserves the relation \sim , that is, if $\omega \sim \omega'$ then $T(\omega) \sim T(\omega')$.

- **(d):** For every $\omega \in \Omega_D$,

$$\alpha(T(\omega)) = \mathbf{1} \oplus \alpha(\omega), \quad \omega \in \Omega_D.$$

- **(e):** For every $n \geq 1$ and $x \in V_n$,

$$T(C_n(i, x)) = C_n(i+1, x), \quad x \in V_n, \quad 0 \leq i < \beta^n - 1.$$

- **(f):** For every $n \geq 1$,

$$\begin{aligned} T(C_n(i)) &= C_n(i+1), \quad 0 \leq i < \beta^n - 1 \\ T(C_n(\beta^n - 1)) &= C_n(0) \end{aligned}$$

Remark. It can be shown that T is the only mapping on Ω_D satisfying the above properties.

3.4 Characterization of Bratteli-Vershik Dynamical Systems

A dynamical system is a quadruple $(\Lambda, \mathcal{F}, P, U)$ in which $(\Lambda, \mathcal{F}, P)$ is a probability space and U is a one-to-one bimeasurable transformation of Λ onto itself. Dynamical system $(\Lambda, \mathcal{F}, P, U)$ is measure-preserving if U preserves the measure P , meaning that $P(E) = P(U(E))$ for every measurable subset E of Λ . Measure-preserving dynamical system $(\Lambda, \mathcal{F}, P, U)$ is ergodic if $P(E) \in \{0, 1\}$ for every event $E \in \mathcal{F}$ in which $U(E) = E$ (such events E are called the invariant events of the system).

For the given β -regular Bratteli diagram D , let $\mathcal{P}(\Omega_D, T)$ be the set of all probability measures P on Ω_D such that T preserves P . For each $P \in \mathcal{P}(\Omega_D, T)$, we have a measure-preserving dynamical system $(\Omega_D, \mathcal{F}(\Omega_D), P, T)$, which is called a *Bratteli-Vershik system* induced by D . We let $\mathcal{P}_e(\Omega_D, T)$ be the collection of all $P \in \mathcal{P}(\Omega_D, T)$ such that the Bratteli-Vershik system $(\Omega_D, \mathcal{F}(\Omega_D), P, T)$ is ergodic. Theorem 3.2 which follows gives (a) a one-to-one correspondence between $\mathcal{S}(D)$ and $\mathcal{P}(\Omega_D, T)$, and (b) a one-to-one correspondence between $\mathcal{S}_e(D)$ and $\mathcal{P}_e(\Omega_D, T)$. In this way, all Bratteli-Vershik systems are characterized in terms of Bratteli-Vershik sources which give rise to them. Theorem 3.2 will be employed in Sec. IV to obtain the ergodic decomposition theorem for Bratteli-Vershik sources.

Theorem 3.2. Let D be the arbitrary β -regular Bratteli diagram fixed at the beginning of this section. For each $\mu \in \mathcal{S}(D)$, there is a unique probability measure P_μ on Ω_D such that

$$P_\mu(C_n[\alpha, x]) = \beta^{-n} \mu_n(x), \quad x \in V_n, \quad \alpha \in S_\beta^n, \quad n \geq 1. \quad (31)$$

The following properties hold:

- (a): If $\mu \in \mathcal{S}(D)$,

$$P_\mu(\{\omega \in \Omega_D : s(\omega_n) = x\}) = \mu_n(x), \quad x \in V_n, \quad n \geq 0. \quad (32)$$

- (b): $\{P_\mu : \mu \in \mathcal{S}(D)\} = \mathcal{P}(\Omega_D, T)$.
- (c): $\{P_\mu : \mu \in \mathcal{S}_e(D)\} = \mathcal{P}_e(\Omega_D, T)$.

We present notation and background needed for our proof of Theorem 3.2. For $n \geq 0$, let $Y_n : \Omega_D \rightarrow E_n$ be the measurable mapping defined by

$$Y_n(\omega) \triangleq \omega_n, \quad \omega \in \Omega_D.$$

For $n \geq 1$, the measurable mapping $Y^{(n)} \triangleq (Y_0, Y_1, \dots, Y_{n-1})$ on Ω_D takes its values in the set $\Pi_D(0, n)$. For $n \geq 0$, the measurable mapping $X_n \triangleq s(Y_n)$ on Ω_D takes its values in V_n . Note that $X_n = r(Y_{n-1})$ for $n \geq 1$. If $n \geq 0$, the measurable mapping $Z_n \triangleq I(Y_n)$ on Ω_D takes its values in S_β . If $n \geq 1$, the measurable mapping

$$Z^{(n)} \triangleq i(Y^{(n)}) = Z_0 + \beta Z_1 + \dots + \beta^{n-1} Z_{n-1}$$

on Ω_D takes its values in $S_{\beta, n}$. For $n \geq 1$,

$$Y^{(n)} = y(Z^{(n)}, X_n)$$

and thus $Y^{(n)}$ and $(Z^{(n)}, X_n)$ are functions of each other. Also,

$$X_n = X_{n+1}[Z_n], \quad n \geq 0,$$

and $(Z^{(n)}, X_n)$ is a function of $(Z^{(n+1)}, X_{n+1})$ for $n \geq 1$. All of the types of cylinder sets defined earlier can be viewed as measurable events involving the measurable functions just defined. In particular, we have

$$C_n(i) = \{Z^{(n)} = i\}$$

$$C_n(i, x) = \{Z^{(n)} = i, X_n = x\} = \{Y^{(n)} = y(i, x)\}$$

$$C_n[\alpha, x] = \{(Z_0, \dots, Z_{n-1}) = \alpha, X_n = x\} = \{Y^{(n)} = y[\alpha, x]\}.$$

Let Y, X, Z be the measurable functions on Ω_D defined by

$$Y \triangleq (Y_0, Y_1, Y_2, \dots)$$

$$X \triangleq (X_0, X_1, X_2, \dots)$$

$$Z \triangleq (Z_0, Z_1, Z_2, \dots)$$

Y is the identity transformation on Ω_D . X takes its values in the Cartesian product measurable space $V_0 \times V_1 \times V_2 \cdots$; its components track the vertices visited along infinite path Y . Z takes its values in \mathbb{Z}_β , and is the address of path Y , that is, $Z = \alpha(Y)$. Later on, placing a certain type of probability measure P on $\mathcal{F}(\Omega_D)$, we will view Y, Z, X as random sequences on probability space $(\Omega_D, \mathcal{F}(\Omega_D), P)$.

We now present two lemmas needed in the proof of Theorem 3.2. We omit the simple proof of the first of these two lemmas.

Lemma 3.3. Let $\mu \in \mathcal{S}(D)$ and let $n \geq 1$. On some probability space, suppose we have a pair $(\tilde{X}_n, \tilde{Z}_{n-1})$ of random objects such that

- \tilde{X}_n is V_n -valued and has probability distribution μ_n .
- Z_{n-1} is S_β -valued and its probability distribution is the uniform distribution on S_β .
- \tilde{X}_n and \tilde{Z}_{n-1} are statistically independent.

Then the V_{n-1} -valued random object $X_n[Z_{n-1}]$ has probability distribution μ_{n-1} .

Lemma 3.4. Let $\mu \in \mathcal{S}(D)$. On some probability space, there exists a random pair (\tilde{X}, \tilde{Z}) such that

- **(a):** \tilde{X} is a random sequence $(\tilde{X}_0, \tilde{X}_1, \dots)$, where \tilde{X}_n is V_n -valued and has probability distribution μ_n ($n \geq 0$).
- **(b):** \tilde{Z} is a random sequence $(\tilde{Z}_0, \tilde{Z}_1, \dots)$, where the \tilde{Z}_n 's are independent and S_β -valued, each uniformly distributed over S_β .
- **(c):** For $n \geq 1$, \tilde{X}_n and $(\tilde{Z}_0, \tilde{Z}_1, \dots, \tilde{Z}_{n-1})$ are independent.
- **(d):** For $n \geq 1$,

$$\Pr(\tilde{X}_{n-1} = \tilde{X}_n[\tilde{Z}_{n-1}]) = 1.$$

Proof. On some probability space, there exists random pair (X^0, Z^0) such that

- X^0 is a random sequence $(X_0^0, X_1^0, X_2^0, \dots)$ with independent components in which X_n^0 is V_n -valued and has probability distribution μ_n ($n \geq 0$).
- Z^0 is a random sequence $(Z_0^0, Z_1^0, Z_2^0, \dots)$ with independent components in which Z_n^0 is S_β -valued and uniformly distributed over S_β ($n \geq 0$).
- X^0, Z^0 are independent.

For each $N \geq 1$, let $X^N = (X_0^N, X_1^N, X_2^N, \dots)$ be the random sequence defined recursively by

$$\begin{aligned} X_n^N &= X_n^0, \quad n \geq N, \\ X_n^N &= X_{n+1}^N[Z_n^0], \quad 0 \leq n < N. \end{aligned}$$

Exploiting Lemma 3.3, we have the properties

- **(e):** For every $N \geq 1$, random pair (X^N, Z^0) obeys the property that X_n^N and $(Z_0^0, Z_1^0, \dots, Z_{n-1}^0)$ are independent for every $n \geq 1$.

- **(f):** For every $N \geq 1$, random sequence X^N obeys the property that component X_n^N of X^N is V_n -valued with probability distribution μ_n for every $n \geq 0$.
- **(g):** For $N > n \geq 0$, $\Pr(X_n^N = X_{n+1}^N | Z_n^0) = 1$.
- **(h):** For each $n \geq 1$, the random $2n$ -tuple

$$(X_0^N, X_1^N, \dots, X_{n-1}^N, Z_0^0, Z_1^0, \dots, Z_{n-1}^0)$$

has probability distribution which does not depend on N for $N \geq n$.

Let measurable space Λ_1 be the Cartesian product space

$$\Lambda_1 = V_0 \times V_1 \times V_2 \times \dots,$$

and let measurable space Λ_2 be the Cartesian product space $\Lambda_1 \times \mathbb{Z}_\beta$. Z^0 is \mathbb{Z}_β -valued, and X^N is Λ_1 -valued for each $N \geq 1$. Thus, the random pair (X^N, Z^0) is Λ_2 -valued for each $N \geq 1$. For each $N \geq 1$, let P_N be the probability measure on Λ_2 which is the probability distribution of (X^N, Z^0) . By property (h), the sequence $\{P_N : N \geq 1\}$ converges weakly to a probability measure P on Λ_2 . Let (\tilde{X}, \tilde{Z}) be any Λ_2 -valued random pair whose probability distribution is P . The sequence of random pairs $\{(X^N, Z^0) : N \geq 1\}$ then converges in distribution to the random pair (\tilde{X}, \tilde{Z}) . As a consequence, \tilde{Z} and Z^0 have the same probability distribution, giving us property (b), and then properties (a),(c),(d) follow by letting $N \rightarrow \infty$ in properties (f),(e),(g), respectively.

Proof of Theorem 3.2 Part 1. Fix $\mu \in S(D)$. We prove there exists unique probability measure P_μ on Ω_D such that (31) holds, and we also prove that (32) holds. On some probability space Λ , we may define processes

$$\tilde{X} = (\tilde{X}_0, \tilde{X}_1, \tilde{X}_2, \dots)$$

$$\tilde{Z} = (\tilde{Z}_0, \tilde{Z}_1, \tilde{Z}_2, \dots)$$

according to Lemma 3.4. As we may throw away a set of measure zero if necessary, we may assume that $\tilde{X}_n = \tilde{X}_{n+1}[\tilde{Z}_n]$ holds everywhere on Λ for every $n \geq 0$. For each $n \geq 0$, let \tilde{Y}_n be the E_n -valued random object $\tilde{Y}_n = e_{\tilde{Z}_n}(\tilde{X}_{n+1})$. This gives us random sequence

$$\tilde{Y} \triangleq (\tilde{Y}_0, \tilde{Y}_1, \tilde{Y}_2, \dots).$$

We have:

- For every $n \geq 0$,

$$r(\tilde{Y}_n) = \tilde{X}_{n+1}, \quad s(\tilde{Y}_n) = \tilde{X}_n, \quad \tilde{Z}_n = I(\tilde{Y}_n)$$

hold everywhere on Λ .

- \tilde{Y} takes its values in $\Pi_D(0, \infty)$.
- The address of \tilde{Y} is \tilde{Z} .

Since \tilde{Z} is a non-degenerate IID process, with probability 1 it takes its values in the set of aperiodic sequences in \mathbb{Z}_β . Therefore, random path \tilde{Y} lies in Ω_D with probability 1. Let P_μ be the probability measure on $\mathcal{F}(\Omega_D)$ such that

$$P_\mu(E) = \Pr\{\tilde{Y} \in E\}, \quad E \in \mathcal{F}(\Omega_D).$$

(Y, X, Z) , regarded as random triple on probability space $(\Omega_D, \mathcal{F}(\Omega_D), P_\mu)$, has the same probability distribution as the random triple $(\tilde{Y}, \tilde{X}, \tilde{Z})$. Evaluating the left side of equation (31), we have

$$\begin{aligned} P_\mu(C_n[\alpha, x]) &= P_\mu\{(Z_0, \dots, Z_{n-1}) = \alpha, X_n = x\} = \\ &= \Pr\{(\tilde{Z}_0, \dots, \tilde{Z}_{n-1}) = \alpha, \tilde{X}_n = x\}. \end{aligned}$$

By independence of $(\tilde{Z}_0, \dots, \tilde{Z}_{n-1})$ and \tilde{X}_n , this last expression factors as

$$\Pr\{(\tilde{Z}_0, \dots, \tilde{Z}_{n-1}) = \alpha\} \Pr\{\tilde{X}_n = x\} = \beta^{-n} \mu_n(x).$$

Thus, (31) holds. P_μ is unique because the cylinder sets generate $\mathcal{F}(\Omega_D)$. Evaluating the left side of (32),

$$P_\mu(\{\omega \in \Omega_D : s(\omega_n) = x\}) = P_\mu\{X_n = x\} = \Pr\{\tilde{X}_n = x\} = \mu_n(x),$$

and thus equation (32) holds.

Proof of Theorem 3.2 Part 2. We prove statement(b) of the theorem. Let $P \in \mathcal{P}(\Omega_D, T)$. For each $n \geq 0$, let μ_n be the PMF on V_n defined by

$$\mu_n(x) \triangleq P\{X_n = x\}, \quad x \in V_n.$$

Let $n \geq 1$ and $x \in V_n$. The cylinder sets $\{C_n(i, x) : 0 \leq i \leq \beta^n - 1\}$ all have the same probability under P because T preserves P and $T^i(C_n(0, x)) = C_n(i, x)$. There are β^n of these cylinder sets and their union is the event $\{X_n = x\}$, so each one must have probability equal to $\beta^{-n} \mu_n(x)$. We have shown

$$P\{Y^{(n)} = y, X_n = x\} = \beta^{-n} \mu_n(x), \quad x \in V_n, \quad y \in \Pi_D(0, n, x), \quad n \geq 1.$$

We prove that

$$\mu_n(x) = \beta^{-1} \sum_{i=0}^{\beta-1} \mu_{n+1}\{w \in V_{n+1} : w[i] = x\}, \quad x \in V_n, \quad n \geq 0. \quad (33)$$

Fix $n \geq 0$ and $x \in V_n$. Let F_n be any event in $\mathcal{F}(\Omega_D)$ (a particular F_n will be chosen later on). Let

$$G = \{(e, w) : e \in E_n, \quad w \in V_{n+1}, \quad s(e) = x, \quad r(e) = w\}.$$

We have

$$P(F_n \cap \{X_n = x\}) = P(F_n \cap \{(Y_n, X_{n+1}) \in G\}),$$

since the events $\{X_n = x\}$ and $\{(Y_n, X_{n+1}) \in G\}$ are identical. For $i \in S_\beta$, let $G_i = \{(e, w) \in G : e = e_i(w)\}$. The G_i 's form a partition of G , and so

$$\begin{aligned} P(F_n \cap \{(Y_n, X_{n+1}) \in G\}) &= \\ \sum_{i=0}^{\beta-1} \sum_{(e,w) \in G_i} P(F_n \cap \{Y_n = e, X_{n+1} = w\}) &= \\ \sum_{i=0}^{\beta-1} \sum_{w \in V_{n+1}} \sum_{e \in G_i[w]} P(F_n \cap \{Y_n = e, X_{n+1} = w\}), \end{aligned}$$

where $G_i[w]$ is the section of G_i at w , namely,

$$G_i[w] = \{e \in E_n : (e, w) \in G_i\}.$$

We have $G_i[w] = \{e_i(w)\}$ if $w \in V_{n+1}$ is such that $s(e_i(w)) = x$, whereas $G_i[w]$ is the empty set if $s(e_i(w)) \neq x$. This fact, coupled with the fact that $s(e_i(w)) = w[i]$, gives us

$$\begin{aligned} P(F_n \cap \{X_n = x\}) &= \\ \sum_{i=0}^{\beta-1} \left\{ \sum_{w \in V_{n+1} : w[i] = x} P(F_n \cap \{Y_n = e_i(w), X_{n+1} = w\}) \right\}. \end{aligned} \quad (34)$$

If $n = 0$, choose $F_n = \Omega_D$, and if $n > 0$, choose F_n to be any cylinder set of form $\{Y^{(n)} = y\}$, where $y \in \Omega_D(0, n, x)$. Whether $n = 0$ or $n > 0$, we have

$$P(F_n \cap \{X_n = x\}) = \beta^{-n} P(\{X_n = x\}) = \beta^{-n} \mu_n(x),$$

$$P(F_n \cap \{Y_n = e_i(w), X_{n+1} = w\}) = \beta^{-(n+1)} \mu_{n+1}(w).$$

Substituting into (34), we have

$$\beta^{-n} \mu_n(x) = \beta^{-n} \beta^{-1} \sum_{i=0}^{\beta-1} \left\{ \sum_{w \in V_{n+1} : w[i] = x} \mu_{n+1}(w) \right\}.$$

Cancelling β^{-n} from both sides, equation (33) has been established. Let $\mu : V \rightarrow [0, 1]$ be the mapping whose restriction to V_n is μ_n for every $n \geq 0$. From (33), we conclude that $\mu \in \mathcal{S}(D)$ and therefore $P = P_\mu$. To finish the proof of Theorem 3.2(b), let $\mu \in \mathcal{S}(D)$ and let $P = P_\mu$; we show $P \in \mathcal{P}(\Omega_D, T)$. Let Q be the probability measure on $\mathcal{F}(\Omega_D)$ such that $Q(F) = P(T(F))$ for $F \in \mathcal{F}(\Omega_D)$. We have to show $P = Q$. For each $n \geq 1$, let \mathcal{F}_n be the finite sub sigma-field of $\mathcal{F}(\Omega_D)$ whose atoms are the $|V_n|(\beta^n - 1)$ cylinder sets

$$C_n(i, x), \quad 0 \leq i < \beta^n - 1, \quad x \in V_n$$

together with the event $C_n(\beta^n - 1)$. For each atom of \mathcal{F}_n of the form $C_n(i, x)$,

$$Q(C_n(i, x)) = P(T(C_n(i, x))) = P(C_n(i + 1, x)) = P(C_n(i, x)),$$

where the rightmost equality is due to the fact that both $P(C_n(i+1, x))$ and $P(C_n(i, x))$ are equal to $\beta^{-n}\mu_n(x)$. Thus, P, Q coincide on each sigma-field \mathcal{F}_n ($n \geq 1$). Let $\mathcal{F}_\infty \triangleq \bigcup_n \mathcal{F}_n$. Since $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for $n \geq 1$, \mathcal{F}_∞ is a field of subsets of Ω_D . P, Q coincide on \mathcal{F}_∞ , and therefore $P = Q$ because the field \mathcal{F}_∞ generates $\mathcal{F}(\Omega_D)$.

Proof of Theorem 3.2 Part 3. We prove statement(c) of the theorem. Let $\mathcal{P}^\dagger = \{P_\mu : \mu \in \mathcal{S}_e(D)\}$. The mapping $\mu \rightarrow P_\mu$ is a one-to-one affine mapping of convex set $\mathcal{S}(D)$ onto convex set $\mathcal{P}(\Omega_D, T)$. Therefore, it maps the set of extreme points of $\mathcal{S}(D)$ onto the set of extreme points of $\mathcal{P}(\Omega_D, T)$. Thus, \mathcal{P}^\dagger is the set of extreme points of $\mathcal{P}(\Omega_D, T)$. By [31, Thm. 2], the set of extreme points of $\mathcal{P}(\Omega_D, T)$ coincides with $\mathcal{P}_e(\Omega_D, T)$. Therefore, $\{P_\mu : \mu \in \mathcal{S}_e(D)\} = \mathcal{P}_e(\Omega_D, T)$ and statement(c) holds.

Remarks. As a by-product of the earlier proofs, we see that a probability measure P on Ω_D belongs to $\mathcal{P}(\Omega_D, T)$ if and only if the pair (X, Z) , regarded as a pair of random sequences on probability space $(\Omega_D, \mathcal{F}(\Omega_D), P)$, satisfies the properties:

- **(a):** Z is an IID random sequence, with each component Z_n uniformly distributed over \mathcal{S}_β .
- **(b):** For each $n \geq 1$, X_n and $(Z_0, Z_1, \dots, Z_{n-1})$ are independent.

Given probability measure P on Ω_D satisfying (a) and (b), then the source $\mu \in \mathcal{S}(D)$ such that $P = P_\mu$ is given by

$$\mu(x) = P\{X_n = x\}, \quad x \in V_n, \quad n \geq 0.$$

Each measure-preserving dynamical system has an entropy, whose definition is reviewed in the next paragraph. Theorem 3.5 below relates the entropies of Bratteli-Vershik dynamical systems to the entropy rates of the Bratteli-Vershik sources that give rise to them. This relationship is needed later on in Sec. V to establish the Shannon-McMillan-Breiman theorem for Bratteli-Vershik sources.

Suppose $(\Lambda, \mathcal{F}, P, U)$ is a measure preserving dynamical system. If \mathcal{W} is a collection of random objects on probability space $(\Lambda, \mathcal{F}, P)$, we let $\sigma(\mathcal{W})$ denote the sub sigma-field of \mathcal{F} generated by \mathcal{W} , that is, $\sigma(\mathcal{W})$ is the smallest sub sigma-field of \mathcal{F} with respect to which all of the RVs in \mathcal{W} are measurable. Let W be an arbitrary random variable on $(\Lambda, \mathcal{F}, P)$ taking finitely many values. For each integer i , let W_i be the random variable $W \circ T^i$. Define

$$H_P(\infty, W) \triangleq \lim_{n \rightarrow \infty} n^{-1} H_P(W_0, \dots, W_{n-1}).$$

(In information-theoretic terms, $H_P(\infty, W)$ is the entropy rate of the random sequence (W_0, W_1, W_2, \dots) .) The entropy of dynamical system $(\Lambda, \mathcal{F}, P, U)$ is defined as

$$H(\Lambda, \mathcal{F}, P, T) \triangleq \sup_W H_P(\infty, W).$$

Furthermore, suppose that W is a generator of the system $(\Lambda, \mathcal{F}, P, U)$, meaning that $\sigma(\{W_i : i \in \mathbb{Z}\}) = \mathcal{F}$. A well known result [32, Cor. 3.12] tells us that

$$H(\Lambda, \mathcal{F}, P, T) = H_P(\infty, W).$$

Theorem 3.5. Let D be the arbitrary β -regular Bratteli diagram fixed at the beginning of this section. For each $\mu \in \mathcal{S}(D)$, the entropy of the Bratteli-Vershik system $(\Omega_D, \mathcal{F}(\Omega_D), P_\mu, T)$ is $H_\infty(\mu)$.

Proof of Theorem 3.5 Part 1. Fix $\mu \in \mathcal{S}(D)$ and let $P = P_\mu$. We show that $H(\Omega_D, \mathcal{F}(\Omega_D), P, T) \leq H_\infty(\mu)$. Let $n \geq 1$. Choose W^n to be a random variable on probability space $(\Omega_D, \mathcal{F}(\Omega_D), P)$ taking $|V_n| + \beta^n - 1$ values such that the atoms of $\sigma(\{W^n\})$ are the $|V_n|$ cylinder sets

$$C_n(\beta^n - 1, x) = \{Z^{(n)} = \beta^n - 1, X_n = x\}, \quad x \in V_n$$

together with the $\beta^n - 1$ cylinder sets

$$C_n(i) = \{Z^{(n)} = i\}, \quad 0 \leq i < \beta^n - 1.$$

For each $i \in \mathbb{Z}$, let W_i^n be the random variable $W^n \circ T^i$. We have measure-preserving dynamical system $(\Omega_D, \mathcal{F}_n, P, T)$ in which $\mathcal{F}_n = \sigma(\{W_i^n : i \in \mathbb{Z}\})$. W^n is a generator for this system, and so

$$H(\Omega_D, \mathcal{F}_n, P, T) = \lim_{m \rightarrow \infty} m^{-1} H_P(W_0^n, W_1^n, \dots, W_{m-1}^n).$$

The terms on the right side of this equation are non-increasing as m increases, and so

$$H(\Omega_D, \mathcal{F}_n, P, T) \leq \beta^{-n} H_P(W_0^n, W_1^n, \dots, W_{\beta^n-1}^n). \quad (35)$$

By properties (e) and (f) of the Vershik transformation, one sees that the atoms of the sigma-field $\sigma(\{W_i^n : i \in S_{n,\beta}\})$ are the $\beta^n |V_n|$ cylinder sets

$$C_n(i, x) = \{Z^{(n)} = i, X_n = x\}, \quad i \in S_{\beta,n}, \quad x \in V_n.$$

The random vector $(W_0^n, W_1^n, \dots, W_{\beta^n-1}^n)$ and the random pair $(Z^{(n)}, X_n)$ are thus functions of each other. We then have

$$\begin{aligned} H_P(W_0^n, W_1^n, \dots, W_{\beta^n-1}^n) &= H_P(Z^{(n)}, X_n) \\ &= H_P(Z^{(n)}) + H_P(X_n) \\ &= \log_2(\beta^n) + H(\mu_n) \end{aligned}$$

Substituting into the right side of (35) and letting $n \rightarrow \infty$, we obtain

$$\overline{\lim}_n H(\Omega_D, \mathcal{F}_n, P, T) \leq \overline{\lim}_n \beta^{-n} [\log_2(\beta^n) + H(\mu_n)] = H_\infty(\mu).$$

We have the monotonicity property

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \dots$$

since $(Z^{(n)}, X_n)$ is a function of $(Z^{(n+1)}, X_{n+1})$ for every $n \geq 1$. In addition to this property, we have the property that the field of sets $\cup_n \mathcal{F}_n$ generates $\mathcal{F}(\Omega_D)$. From these properties, [32, Thm. 5.10] tells us that the sequence $\{H(\Omega_D, \mathcal{F}_n, P, T) : n \geq 1\}$ is non-decreasing and

$$H(\Omega_D, \mathcal{F}(\Omega_D), P, T) = \lim_{n \rightarrow \infty} H(\Omega_D, \mathcal{F}_n, P, T).$$

We conclude that $H(\Omega_D, \mathcal{F}(\Omega_D), P, T) \leq H_\infty(\mu)$.

Proof of Theorem 3.5 Part 2. We show that

$$H_\infty(\mu) \leq H(\Omega_D, \mathcal{F}(\Omega_D), P_\mu, T), \quad \mu \in \mathcal{S}(D).$$

For convenience, we assume that the diagram $D = (V, E)$ is canonical in this proof. (There is no loss of generality in assuming this, since a regular diagram is isomorphic to a canonical one.) Thus, V_n is a subset of $V_0^{\beta^n}$ for every $n \geq 1$. Letting $n \geq 1$ and $x \in V_n$, we have the factorization

$$x = x[0]x[1] \cdots x[\beta - 1],$$

in which the $x[i]$'s belong to V_{n-1} . (The indexing $I : E \rightarrow S_\beta$ is assumed to be the natural one in which the edge $e_i(x)$ such that $I(e_i(x)) = i$ has source vertex $x[i] \in V_{n-1}$.) Fix $\mu \in \mathcal{S}(D)$ and let $P = P_\mu$. The sequences $X = (X_n : n \geq 0)$ and $Z = (Z_n : n \geq 0)$ of measurable functions on Ω_D defined earlier are regarded as random sequences on probability space $(\Omega_D, \mathcal{F}(\Omega_D), P)$ in this proof. For each integer i , let W_i be the V_0 -valued random object $W_i = X_0 \circ T^i$. We have

$$\lim_{m \rightarrow \infty} m^{-1} H(W_0, W_1, \dots, W_{m-1}) \leq H(\Omega_D, \mathcal{F}(\Omega_D), P, T).$$

Fix arbitrary positive integer m . Fix arbitrary positive integer n such that $\beta^n > m$. Since $V_n \subset V_0^{\beta^n}$ and X_n is V_n -valued, X_n is a random β^n -tuple, which we write as

$$X_n = (U_0, U_1, \dots, U_{\beta^n-1}),$$

where coordinate U_i of X_n is a V_0 -valued random object. Using the formula

$$X_{j-1} = X_j[Z_{j-1}], \quad 1 \leq j \leq n,$$

it follows that

$$\{Z^{(n)} = i\} \subset \{X_0 = U_i\}, \quad 0 \leq i \leq \beta^n - 1. \quad (36)$$

Let

$$R_j = \{0 \leq i \leq \beta^n - m : \text{mod}(i, m) = j\}, \quad j \in \{0, 1, \dots, m-1\}.$$

Fix $j \in \{0, 1, \dots, m-1\}$. The blocks $\{U_i^{i+m-1} : i \in R_j\}$ are adjacent non-overlapping blocks of length m in X_n which together constitute all of X_n except for at most $m-1$ coordinates of X_n at the beginning and at most $m-1$ coordinates of X_n at the end. Each coordinate U_i of X_n has entropy at most $\log_2 |V_0|$. Thus, by subadditivity of entropy,

$$H(\mu_n) = H_P(X_n) \leq 2(m-1)\log_2 |V_0| + \sum_{i \in R_j} H_P(U_i^{i+m-1}).$$

Summing over j , we obtain

$$mH(\mu_n) \leq 2m(m-1)\log_2 |V_0| + \sum_{i=0}^{\beta^n-m} H_P(U_i^{i+m-1}). \quad (37)$$

Suppose $Z^{(n)} = i$, where $0 \leq i \leq \beta^n - m$. Let $j \in \{0, \dots, m-1\}$. Then $0 \leq i+j \leq \beta^n - 1$ and so by properties of the Vershik transformation, we have $Z^{(n)} \circ T^j = i+j$ and $X_n \circ T^j = X_n$. By (36), $W_j = X_0 \circ T^j$ is equal to coordinate $i+j$ of $X_n \circ T^j$, which is U_{i+j} , coordinate $i+j$ of X_n . We have shown that

$$\{Z^{(n)} = i\} \subset \{W_0^{m-1} = U_i^{i+m-1}\}, \quad 0 \leq i \leq \beta^n - m,$$

which implies

$$H_P(W_0^{m-1} | Z^{(n)} = i) = H_P(U_i^{i+m-1} | Z^{(n)} = i), \quad 0 \leq i \leq \beta^n - m.$$

For each such i , the random m -tuple U_i^{i+m-1} and $Z^{(n)}$ are statistically independent, because U_i^{i+m-1} is a function of X_n and $X_n, Z^{(n)}$ are independent. Thus,

$$H_P(U_i^{i+m-1} | Z^{(n)} = i) = H_P(U_i^{i+m-1}), \quad 0 \leq i \leq \beta^n - m.$$

Employing (37),

$$\begin{aligned} \sum_{i=0}^{\beta^n - m} H_P(W_0^{m-1} | Z^{(n)} = i) &= \sum_{i=0}^{\beta^n - m} H_P(U_i^{i+m-1}) \\ &\geq mH(\mu_n) - 2m(m-1)\log_2 |V_0|. \end{aligned} \quad (38)$$

We also have

$$\begin{aligned} H_P(W_0^{m-1}) &\geq H_P(W_0^{m-1} | Z^{(n)}) \\ &= \sum_{i=0}^{\beta^n - 1} P\{Z^{(n)} = i\} H_P(W_0^{m-1} | Z^{(n)} = i) \\ &\geq \beta^{-n} \sum_{i=0}^{\beta^n - m} H_P(W_0^{m-1} | Z^{(n)} = i), \end{aligned}$$

using the fact that $Z^{(n)}$ is equiprobable over the set $\{0, 1, \dots, \beta^n - 1\}$. Combining with (38), we have shown that

$$m^{-1} H_P(W_0^{m-1}) \geq \beta^{-n} H(\mu_n) - 2\beta^{-n}(m-1)\log_2 |V_0|$$

holds for any pair (m, n) of positive integers for which $\beta^n > m$. Holding m fixed and letting $n \rightarrow \infty$ on both sides of this statement,

$$m^{-1} H_P(W_0^{m-1}) \geq H_\infty(\mu), \quad m \geq 1.$$

Letting $m \rightarrow \infty$,

$$H(\Omega_D, \mathcal{F}(\Omega_D), P, T) \geq \lim_{m \rightarrow \infty} m^{-1} H_P(W_0^{m-1}) \geq H_\infty(\mu), \quad (39)$$

and our proof is complete.

Remark. Letting $W_j = X_0 \circ T^j$ for $j \in \mathbb{Z}$, we see that

$$H(\Omega_D, \mathcal{F}(\Omega_D), P_\mu, T) = H_\infty(\mu) = \lim_{m \rightarrow \infty} m^{-1} H_{P_\mu}(W_0^{m-1})$$

holds for every $\mu \in \mathcal{P}(D)$, because by Theorem 3.5 the rightmost and leftmost quantities in (39) are equal.

4 Decomposition Theorems

In this section, we prove the Ergodic Decomposition Theorem (Theorem 1.7) and the Entropy Rate Decomposition Theorem (Theorem 1.8). We prove Theorem 1.7 first. Our proof of Theorem 1.7 employs the following representation theorem, proved in [31].

Blum-Hanson Representation Theorem. Let (Γ, \mathcal{F}) be a measurable space, and let U be a one-to-one bimeasurable transformation of Γ onto itself. Let \mathcal{P} be the set of probability measures on Γ which are preserved by U , let \mathcal{F}_U be the set of events in \mathcal{F} which are U -invariant, and let $\mathcal{P}_e = \{P \in \mathcal{P} : P(\mathcal{F}_U) = \{0, 1\}\}$ (the set of measures in \mathcal{P} which are ergodic with respect to U). Suppose that

$$\begin{aligned} \{F \in \mathcal{F}_U : P(F) = 0 \text{ for all } P \in \mathcal{P}_e\} &= \\ \{F \in \mathcal{F}_U : P(F) = 0 \text{ for all } P \in \mathcal{P}\} \end{aligned} \quad (40)$$

Let $\mathcal{F}(\mathcal{P}_e)$ be the smallest sigma-field of subsets of \mathcal{P}_e such that for each $F \in \mathcal{F}$, the mapping $P \rightarrow P(F)$ is a measurable mapping from \mathcal{P}_e into \mathbb{R} . Then for each $P \in \mathcal{P}$, there exists a unique probability measure λ_P on $\mathcal{F}(\mathcal{P}_e)$ such that

$$P(F) = \int_{\mathcal{P}_e} Q(F) d\lambda_P(Q), \quad F \in \mathcal{F}.$$

We proceed with the proof of Theorem 1.7 after establishing Lemmas 4.1-4.2 below. Throughout the rest of this section, $D = (V, E)$ is a fixed β -regular Bratteli diagram. We need the following definitions.

Definitions.

- For each $v \in V$, $\Phi_v : \mathcal{S}_e(D) \rightarrow \mathbb{R}$ is the mapping defined by

$$\Phi_v(\mu) \triangleq \mu(v), \quad \mu \in \mathcal{S}_e(D).$$

- For each $G \in \mathcal{F}(\Omega_D)$, $\Phi_G : \mathcal{P}_e(\Omega_D, T) \rightarrow \mathbb{R}$ is the mapping defined by

$$\Phi_G(P) \triangleq P(G), \quad P \in \mathcal{P}_e(\Omega_D, T).$$

- Define $\mathcal{F}(\mathcal{P}_e(\Omega_D, T))$ to be the smallest of all sigma-fields \mathcal{F} of subsets of $\mathcal{P}_e(\Omega_D, T)$ such that Φ_G is \mathcal{F} -measurable for each $G \in \mathcal{F}(\Omega_D)$.
- Define the mapping $\Psi : \mathcal{S}_e(D) \rightarrow \mathcal{P}_e(\Omega_D, T)$ by

$$\Psi(\mu) \triangleq P_\mu, \quad \mu \in \mathcal{S}_e(D).$$

Recall that in Sec. I, $\mathcal{S}_e(D)$ was taken to be the measurable space with sigma-field the family of all Borel subsets with respect to the topology on $\mathcal{S}_e(D)$. Let $\mathcal{F}(\mathcal{S}_e(D))$ denote this sigma-field of Borel sets. $\mathcal{F}(\mathcal{S}_e(D))$ is then the smallest sigma-field of subsets of $\mathcal{S}_e(D)$ containing all $\mathcal{S}_e(D)$ -open sets.

Lemma 4.1. $\mathcal{F}(\mathcal{S}_e(D))$ is the smallest of all sigma-fields \mathcal{F} of subsets of $\mathcal{S}_e(D)$ such that the mapping Φ_v is \mathcal{F} -measurable for every $v \in V$.

Proof. Let \mathcal{F}_1 be the smallest of all sigma-fields \mathcal{F} of subsets of $\mathcal{S}_e(D)$ such that the mapping Φ_v is \mathcal{F} -measurable for every $v \in V$. Part 1 of the proof is to show that $\mathcal{F}_1 \subset \mathcal{F}(\mathcal{S}_e(D))$. Part 2 of the proof is to show that $\mathcal{F}(\mathcal{S}_e(D)) \subset \mathcal{F}_1$.

Part 1 of Proof. $\mathcal{F}_1 \subset \mathcal{F}(\mathcal{S}_e(D))$ is established by showing that every Φ_v is $\mathcal{F}(\mathcal{S}_e(D))$ -measurable. Fix Φ_v and any real number x . We have to show that the set $S = \Phi_v^{-1}((-\infty, x))$ belongs to $\mathcal{F}(\mathcal{S}_e(D))$. Since Φ_v is continuous with respect to the topology on $\mathcal{S}_e(D)$, the set S is $\mathcal{S}_e(D)$ -open. But then $S \in \mathcal{F}(\mathcal{S}_e(D))$ because this sigma-field contains all $\mathcal{S}_e(D)$ -open sets.

Part 2 of Proof. $\mathcal{F}(\mathcal{S}_e(D)) \subset \mathcal{F}_1$ is established by showing that \mathcal{F}_1 contains every $\mathcal{S}_e(D)$ -open set. Let \mathcal{B} be the collection of all subsets of $\mathcal{S}_e(D)$ of the form

$$\bigcap_{v \in S} \Phi_v^{-1}(I_v),$$

where S is a finite subset of V and each I_v is a non-empty bounded open sub-interval of the real line having rational endpoints. $\mathcal{S}_e(D)$'s topology is the set of all unions of members of \mathcal{B} . Every member of \mathcal{B} belongs to \mathcal{F}_1 , and \mathcal{B} is countable. If O is an arbitrary $\mathcal{S}_e(D)$ -open set, we argue that $O \in \mathcal{F}_1$ as follows: let $\{O_i\}$ be a subset of \mathcal{B} such that $O = \bigcup_i O_i$, which is a countable union of members of \mathcal{F}_1 since \mathcal{B} is countable. O must therefore belong to \mathcal{F}_1 , since any sigma-field is closed with respect to countable unions.

Lemma 4.2. The mapping Ψ is a one-to-one bimeasurable mapping of measurable space $(\mathcal{S}_e(D), \mathcal{F}(\mathcal{S}_e(D)))$ onto measurable space $(\mathcal{P}_e(\Omega_D, T), \mathcal{F}(\mathcal{P}_e(\Omega_D, T)))$.

Proof. Let $\mathcal{F}_1 = \mathcal{F}(\mathcal{S}_e(D))$ and let $\mathcal{F}_2 = \mathcal{F}(\mathcal{P}_e(\Omega_D, T))$. From our previous results, Ψ is a one-to-one mapping of $\mathcal{S}_e(D)$ onto $\mathcal{P}_e(\Omega_D, T)$. What remains is to prove that Ψ is bimeasurable. That is, we have to prove the two statements

$$\Psi^{-1}(\mathcal{F}_2) \subset \mathcal{F}_1 \quad (41)$$

$$\Psi(\mathcal{F}_1) \subset \mathcal{F}_2 \quad (42)$$

This gives us two parts of the proof.

Part 1: Proof of (41). Define the sigma-field \mathcal{F}_2^* of subsets of $\mathcal{P}_e(\Omega_D, T)$ by

$$\mathcal{F}_2^* \triangleq \{F \in \mathcal{F}_2 : \Psi^{-1}(F) \in \mathcal{F}_1\}.$$

If we can show that $\mathcal{F}_2 \subset \mathcal{F}_2^*$, then we will have $\mathcal{F}_2 = \mathcal{F}_2^*$, establishing (41). By definition of \mathcal{F}_2 , it will follow that $\mathcal{F}_2 \subset \mathcal{F}_2^*$ if we can show that the mapping Φ_G is \mathcal{F}_2^* -measurable for every $G \in \mathcal{F}(\Omega_D)$. First, we consider the case when G is the cylinder set $G = C_n(i, v)$, where $n \geq 1$, $i \in S_{\beta, n}$, and $v \in V_n$ are arbitrary. To show that Φ_G is \mathcal{F}_2^* -measurable, we must show that the set $S_x = \Phi_G^{-1}((-\infty, x)) \in \mathcal{F}_2^*$ for every $x \in \mathbb{R}$, which by definition of \mathcal{F}_2^* means we must show that $\Psi^{-1}(S_x) \in \mathcal{F}_1$. We have

$$\Psi^{-1}(S_x) = \{\mu \in \mathcal{S}_e(D) : P_\mu(G) < x\} = \{\mu \in \mathcal{S}_e(D) : \mu(v) < \beta^n x\},$$

using the fact that $P_\mu(C_n(i, v)) = \beta^{-n} \mu(v)$. Thus, $\Psi^{-1}(S_x) \in \mathcal{F}_1$ because it is an $\mathcal{S}_e(D)$ -open set. Let

$$\mathcal{M} = \{G \in \mathcal{F}(\Omega_D) : \Phi_G \text{ is } \mathcal{F}_2^* \text{-measurable}\}.$$

\mathcal{M} is a monotone class containing the field generated by the cylinder sets. By the monotone class theorem [33, Thm. 1.3.9], $\mathcal{M} = \mathcal{F}(\Omega_D)$. We conclude that Φ_G is \mathcal{F}_2^* -measurable for every $G \in \mathcal{F}(\Omega_D)$, completing Part 1.

Part 2: Proof of (42). Define the sigma-field \mathcal{F}_1^* of subsets of $\mathcal{S}_e(D)$ by

$$\mathcal{F}_1^* \triangleq \{F \in \mathcal{F}_1 : \Psi(F) \in \mathcal{F}_2\}.$$

If we can show that $\mathcal{F}_1 \subset \mathcal{F}_1^*$, then we will have $\mathcal{F}_1 = \mathcal{F}_1^*$, establishing (42). By Lemma 4.1, it will follow that $\mathcal{F}_1 \subset \mathcal{F}_1^*$ if we can show that the mapping Φ_v is \mathcal{F}_1^* -measurable for every $v \in V$. Fix $v \in V$. Then $v \in V_n$ for some $n \geq 0$. Let $G = \{X_n = v\}$. Let $x \in \mathbb{R}$ be arbitrary. Then

$$\Psi(\Phi_v^{-1}((-\infty, x))) = \Phi_G^{-1}((-\infty, x)) \in \mathcal{F}_2,$$

and so the event $\Phi_v^{-1}((-\infty, x))$ belongs to \mathcal{F}_1^* . We conclude that Φ_v is \mathcal{F}_1^* -measurable, completing Part 2.

4.1 Proof of Theorem 1.7

In this proof, we use the notations $\mathcal{P}(T)$ and $\mathcal{P}_e(T)$ to denote the spaces $\mathcal{P}(\Omega_D, T)$ and $\mathcal{P}_e(\Omega_D, T)$, respectively. As before, let $\mathcal{F}_1 = \mathcal{F}(\mathcal{S}_e(D))$ and let $\mathcal{F}_2 = \mathcal{F}(\mathcal{P}_e(\Omega_D, T))$. Recall that \mathcal{F}_T is the sub sigma-field of $\mathcal{F}(\Omega_D)$ consisting of the T -invariant sets. We have broken down the proof of Theorem 1.7 into three parts.

Part 1. Let $P \in \mathcal{P}(T)$. We show there exists probability measure τ_P on measurable space $(\mathcal{P}_e(T), \mathcal{F}_2)$ which represents P in the sense that

$$P(F) = \int_{\mathcal{P}_e(T)} Q(F) d\tau_P(Q), \quad F \in \mathcal{F}(\Omega_D). \quad (43)$$

Let $\mu \in \mathcal{S}(D)$ be the measure such that $P_\mu = P$. By Lemma 1.1, there exists probability measure λ on measurable space $(\mathcal{S}_e(D), \mathcal{F}_1)$ such that

$$\mu(v) = \int_{\mathcal{S}_e(D)} \sigma(v) d\lambda(\sigma), \quad v \in V. \quad (44)$$

From this it follows that

$$P(F) = \int_{\mathcal{S}_e(D)} P_\sigma(F) d\lambda(\sigma) \quad (45)$$

for every cylinder set

$$F = C_n(i, v), \quad i \in S_{\beta, n}, \quad v \in V_n, \quad n \geq 1.$$

(Multiply both sides of (44) by β^{-n} and then use the facts that $P(F) = \beta^{-n}\mu(v)$ and $P_\sigma(F) = \beta^{-n}\sigma(v)$.) The collection of events $F \in \mathcal{F}(\Omega_D)$ for which equation (45) holds is a monotone class containing the field generated by the cylinder sets, and thus (45) holds for every $F \in \mathcal{F}(\Omega_D)$ by the monotone class theorem [33, Thm. 1.3.9]. By Lemma 4.2, we have probability measure τ_P on $(\mathcal{P}_e(T), \mathcal{F}_2)$ defined by

$$\tau_P(G) \triangleq \lambda(\Psi^{-1}(G)) = \lambda(\{\mu \in \mathcal{S}_e(D) : P_\mu \in G\}), \quad G \in \mathcal{F}_2.$$

Making a change of variable in (45), we obtain (43).

Part 2. For each $P \in \mathcal{P}(T)$, we show that τ_P representing P in the sense of (43) is unique. Suppose $G \in \mathcal{F}_T$ and $Q(G) = 0$ for all $Q \in \mathcal{P}_e(T)$. Fix arbitrary $P \in \mathcal{P}(T)$. By Part 1, measure τ_P on $\mathcal{P}_e(T)$ exists such that (43) holds. Setting $F = G$ in this equation, we conclude that $P(G) = 0$. We have verified that property (40) holds for the dynamical system $(\Lambda, \mathcal{F}, U) = (\Omega_D, \mathcal{F}(\Omega_D), T)$. Thus, for each $P \in \mathcal{P}(T)$, the representing measure τ_P in (43) is unique by the Blum-Hanson Representation Theorem.

Part 3. Let $\mu \in \mathcal{S}(D)$. To complete the proof of Theorem 1.7, we have to show that the measure λ on $(\mathcal{S}_e(D), \mathcal{F}_1)$ satisfying (44) is unique. That is, we must show $\lambda_1 = \lambda_2$, where λ_1 and λ_2 are any two probability measures on $(\mathcal{S}_e(D), \mathcal{F}_1)$ for which

$$\mu(v) = \int_{\mathcal{S}_e(D)} \sigma(v) d\lambda_1(\sigma) = \int_{\mathcal{S}_e(D)} \sigma(v) d\lambda_2(\sigma), \quad v \in V. \quad (46)$$

Let $P = P_\mu$. Let τ_1, τ_2 be the probability measures on $(\mathcal{P}_e(T), \mathcal{F}_2)$ such that

$$\tau_i(G) \triangleq \lambda_i(\Psi^{-1}(G)), \quad G \in \mathcal{F}_2, \quad i = 1, 2.$$

As argued in Part 1,

$$P(F) = \int_{\mathcal{P}_e(T)} Q(F) d\tau_1(Q) = \int_{\mathcal{P}_e(T)} Q(F) d\tau_2(Q), \quad F \in \mathcal{F}(\Omega_D).$$

By Part 2, $\tau_1 = \tau_2$. Let $G \in \mathcal{F}_1$ be arbitrary. Let $G' = \Psi(G)$, which is an event in \mathcal{F}_2 by Lemma 4.2. Since $G = \Psi^{-1}(G')$, we have

$$\tau_i(G') = \lambda_i(\Psi^{-1}(G')) = \lambda_i(G), \quad i = 1, 2.$$

Therefore, $\lambda_1(G) = \lambda_2(G)$ since $\tau_1(G') = \tau_2(G')$. We conclude that $\lambda_1 = \lambda_2$, completing the proof of Theorem 1.7.

4.2 Proof of Theorem 1.8

Throughout this subsection, we have a fixed β -regular Bratteli diagram $D = (V, E)$. By Theorem 1.5, there exists for each $n \geq 0$ a lossless prefix encoder $\phi_n : V_n \rightarrow \{0, 1\}^*$ such that

$$\lim_{n \rightarrow \infty} \beta^{-n} \bar{L}(\phi_n, \sigma_n) = H_\infty(\sigma), \quad \sigma \in \mathcal{S}(D).$$

For each $n \geq 0$, let $\psi_n : V_n \rightarrow \{0, 1\}^*$ be a lossless prefix encoder employing fixed-length codewords of length $\lceil \log_2 |V_n| \rceil$. For each $n \geq 0$, let $\phi_n^* : V_n \rightarrow \{0, 1\}^*$ be the lossless prefix encoder $\phi_n^* = \phi_n \wedge \psi_n$. For $\sigma \in \mathcal{S}(D)$ and $n \geq 0$, we have

$$\beta^{-n} H(\sigma_n) \leq \beta^{-n} \bar{L}(\phi_n^*, \sigma_n) \leq \beta^{-n} (\bar{L}(\phi_n, \sigma_n) + 1).$$

Therefore,

$$\lim_{n \rightarrow \infty} \beta^{-n} \bar{L}(\phi_n^*, \sigma_n) = H_\infty(\sigma), \quad \sigma \in \mathcal{S}(D).$$

Since $|V_n| \leq |V_0|^{\beta^n}$,

$$\beta^{-n} \bar{L}(\phi_n^*, \sigma_n) \leq \beta^{-n} (\lceil \log_2 |V_n| \rceil + 1) \leq \log_2 |V_0| + 2$$

holds for all $\sigma \in \mathcal{S}(D)$ and $n \geq 0$.

Fix $\mu \in \mathcal{S}(D)$. Let λ_μ be the unique probability measure on $\mathcal{S}_e(D)$ such that

$$\mu(x) = \int_{\mathcal{S}_e(D)} \sigma(x) d\lambda_\mu(\sigma), \quad x \in V.$$

Then

$$\beta^{-n} \bar{L}(\phi_n^*, \mu_n) = \int_{\mathcal{S}_e(D)} \beta^{-n} \bar{L}(\phi_n^*, \sigma_n) d\lambda_\mu(\sigma), \quad n \geq 0. \quad (47)$$

By the bounded convergence theorem of probability theory,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathcal{S}_e(D)} \beta^{-n} \bar{L}(\phi_n^*, \sigma_n) d\lambda_\mu(\sigma) &= \\ \int_{\mathcal{S}_e(D)} \left[\lim_{n \rightarrow \infty} \beta^{-n} \bar{L}(\phi_n^*, \sigma_n) \right] d\lambda_\mu(\sigma) &= \int_{\mathcal{S}_e(D)} H_\infty(\sigma) d\lambda_\mu(\sigma). \end{aligned}$$

Thus, letting $n \rightarrow \infty$ on both sides of equation (47), we obtain

$$H_\infty(\mu) = \int_{\mathcal{S}_e(D)} H_\infty(\sigma) d\lambda_\mu(\sigma),$$

the desired conclusion of Theorem 1.8.

5 Shannon-McMillan-Breiman Theorem

This section proves the Shannon-McMillan-Breiman theorem (SMB theorem) for Bratteli-Vershik information sources. We first review the SMB theorem for stationary finite-alphabet sequential sources. Let A be a finite non-empty set. Let $\{U_i : i \geq 0\}$ be a stationary random sequence of A -valued random objects defined on a probability space Λ . For $n \geq 1$, let p_n be the PMF on A^n which is the probability distribution of the A^n -valued random vector (U_1, \dots, U_n) . The SMB theorem for $\{U_i\}$ states that the sequence of random variables $\{-n^{-1} \log_2 p_n(U_1, \dots, U_n) : n \geq 1\}$ converges in $L^1[\Lambda]$ space norm and almost surely to a random variable on Λ belonging to $L^1[\Lambda]$. $L^1[\Lambda]$ convergence in the SMB theorem is due to Shannon and McMillan [4]. Almost sure convergence in the SMB theorem is due to Breiman [5].

Let $D = (V, E)$ be a β -regular Bratteli diagram, fixed throughout this section. Recall the collections $\{Y_n : n \geq 0\}$, $\{X_n : n \geq 0\}$, and $\{Y^{(n)} : n \geq 1\}$ of measurable functions on Ω_D defined in Section III. Let $\mu \in \mathcal{S}(D)$ be a fixed Bratteli-Vershik information source on D and let $P = P_\mu$, which is the probability measure on $\mathcal{F}(\Omega_D)$ such that

$$P\{Y^{(n)} = y, X_n = x\} = \beta^{-n} \mu_n(x), \quad x \in V_n, \quad y \in \Pi_D(0, n, x), \quad n \geq 1.$$

This gives us the probability space $\Lambda_P = (\Omega_D, \mathcal{F}(\Omega_D), P)$. The expected value of a random variable U on Λ_P shall be denoted $E_P[U]$. Let $L^1[\Lambda_P]$ be the space of random variables U on Λ_P for which $E_P[|U|] < \infty$. For each $n \geq 0$, the random object $X_n : \Lambda_P \rightarrow V_n$ has probability distribution μ_n . Furthermore, the random variable $-\log_2 \mu_n(X_n)$

belongs to $L^1[P]$ and its expected value is $H(\mu_n)$. The SMB theorem for Bratteli-Vershik sources (Theorem 5.3 below) tells us that the sequence of random variables $\{-\beta^{-n} \log_2 \mu_n(X_n) : n \geq 0\}$ converges in $L^1[\Lambda_P]$ and almost surely $[P]$ to some limit function in $L^1[\Lambda_P]$, and moreover, the limit function and its distribution are identified. To give the full statement of this theorem, we need to lay some groundwork below.

Let us recall some of the machinery from Section III. Fix an indexing $I : E \rightarrow S_\beta$. For each $n \geq 1$ and $x \in V_n$ and $i \in S_\beta$, we have the vertex $x[i] \in V_{n-1}$ which is the source vertex of the edge in $E(x)$ whose I -value is i . Letting $Z_n \triangleq I(Y_n)$, we obtain the IID sequence $Z = \{Z_n : n \geq 0\}$ of random variables on Λ_P in which each Z_n is S_β -valued and is uniformly distributed over S_β . For $n \geq 1$, $(Z_0, Z_1, \dots, Z_{n-1})$ and X_n are statistically independent random objects on Λ_P and $X_{n-1} = X_n[Z_{n-1}]$ holds everywhere on Ω_D . Furthermore, letting

$$Z^{(n)} = Z_0 + Z_1\beta + \dots + Z_{n-1}\beta^{n-1}, \quad n \geq 1,$$

$Z^{(n)}$ is uniformly distributed over $S_{\beta,n}$.

N is the positive integer valued random variable on Λ_P defined in Sec. III by

$$N \triangleq \min\{n \geq 1 : Z_{n-1} < \beta - 1\}.$$

We have $N = n$ if and only if the path $Y^{(n)} \in \Pi_D(0, n)$ is not a final path and no prefix of $Y^{(n)}$ is a final path. The random variable N gave rise to the Vershik transformation in Sec. III: if $N(\omega) = n$, then $T(\omega)_0^{n-1} = T_n(\omega_0^{n-1})$ and $T(\omega)_n^\infty = \omega_n^\infty$. N has a geometric distribution:

$$P\{N = n\} = (\beta^{-1})^{n-1}(1 - \beta^{-1}), \quad n \geq 1.$$

Another useful fact about random variable N is that it is a stopping time relative to the random sequence $Z = (Z_i : i \geq 0)$, that is, the event $\{N = n\}$ is (Z_0, \dots, Z_{n-1}) -measurable for every $n \geq 1$.

Lemma 5.1. Let $\mu \in \mathcal{S}(D)$, let $P = P_\mu$, let Λ_P be the probability space $(\Omega_D, \mathcal{F}(\Omega_D), P)$, and let $n \geq 1$. Then

$$\{x \in V_n : \mu(x) > 0\} = \bigcap_{i=0}^{\beta-1} \{x \in V_n : \mu(x) > 0, \mu(x[i]) > 0\}, \quad (48)$$

and for the resulting $L^1[\Lambda_P]$ random variable

$$W_n \triangleq \log_2 \frac{\mu_n(X_n)}{\prod_{i=0}^{\beta-1} \mu_{n-1}(X_n[i])}, \quad (49)$$

we have

$$\begin{aligned} E_P[W_n] &= \beta H(\mu_{n-1}) - H(\mu_n) \\ E_P[|W_n|] &\leq 2e^{-1} \log_2 e + \beta H(\mu_{n-1}) - H(\mu_n) \end{aligned}$$

Proof. Fix μ and let $P = P_\mu$. This gives us probability space Λ_P . Let $n \geq 1$. Let $x \in V_n$ satisfy $\mu_n(x) > 0$. Let $i \in S_\beta$ and let $u = x[i]$. We show $\mu_{n-1}(u) > 0$, establishing

(48). We have

$$\mu_{n-1}(u) = \beta^{-1} \sum_{j=0}^{\beta-1} \mu_n(\{x' \in V_n : x'[j] = u\}). \quad (50)$$

Since $x \in \{x' \in V_n : x'[i] = u\}$, the μ_n -measure of this set is > 0 and hence $\mu_{n-1}(u) > 0$. Recall that X_m has PMF μ_m for every $m \geq 0$. Hence,

$$P\{\mu_n(X_n) > 0\} = \mu_n(\{x \in V_n : \mu_n(x) > 0\}) = 1,$$

and so $\log_2 \mu_n(X_n)$ belongs to $L^1[\Lambda_P]$. By (48),

$$P\{\mu_{n-1}(X_n[i]) > 0\} = \mu_n(\{x \in V_n : \mu_{n-1}(x[i]) > 0\}) = 1, \quad i \in S_\beta,$$

and so $\log_2 \mu_{n-1}(X_n[i])$ belongs to $L^1[\Lambda_P]$ for $i \in S_\beta$. It follows that W_n belongs to $L^1[\Lambda_P]$ because

$$W_n = \log_2 \mu_n(X_n) - \sum_{i=0}^{\beta-1} \log_2(X_n[i]).$$

Since (50) holds for every $u \in V_{n-1}$, it follows that

$$E_P[f(X_{n-1})] = \beta^{-1} \sum_{i=0}^{\beta-1} E_P[f(X_n[i])]$$

for every extended real-valued function $f : V_{n-1} \rightarrow [0, \infty]$. Taking f to be the function

$$f(x) \triangleq -\log_2 \mu_{n-1}(x), \quad x \in V_{n-1},$$

we have

$$\begin{aligned} E_P[W_n] &= E_P[\log_2 \mu_n(X_n)] + \sum_{i=0}^{\beta-1} E_P[f(X_n[i])] \\ &= E_P[\log_2 \mu_n(X_n)] + \beta E_P[f(X_{n-1})] \\ &= -H(\mu_n) + \beta H(\mu_{n-1}) \end{aligned}$$

The proof is completed by showing that

$$E_P[|W_n|] \leq 2e^{-1} \log_2 e + E_P[W_n] \quad (51)$$

As discussed in Sec. I, there is an isomorphism carrying $D = (V, E)$ into a canonical diagram $\tilde{D} = (\tilde{V}, \tilde{E})$ in which every \tilde{V}_n is a subset of $V_0^{\beta^n}$. Under this isomorphism, for each $n \geq 0$ and $x \in V_n$, there corresponds a string $\eta(x)$ in $V_0^{\beta^n}$. Furthermore, for $n \geq 1$ and $x \in V_n$,

$$\eta(x) = \eta(x[0])\eta(x[1]) \cdots \eta(x[\beta-1]).$$

On probability space Λ_P , we have the collections of random objects $\{\tilde{X}_n : n \geq 0\}$ and $\{\tilde{X}_n[i] : n \geq 1, i \in S_\beta\}$ such that

- \tilde{X}_n is the $V_0^{\beta^n}$ -valued random object $\eta(X_n)$.
- $\tilde{X}_n[i]$ is the $V_0^{\beta^{n-1}}$ -valued random object $\eta(X_n[i])$.

For $n \geq 0$, let $\tilde{\mu}_n$ be the PMF of \tilde{X}_n on $V_0^{\beta^n}$. For $n \geq 1$, let λ_n be the β -fold product PMF on $V_0^{\beta^n}$ defined by

$$\lambda_n \triangleq \tilde{\mu}_{n-1} \times \tilde{\mu}_{n-1} \times \cdots \times \tilde{\mu}_{n-1}.$$

Fix $n \geq 1$. Since

$$\begin{aligned} \mu_n(X_n) &= \tilde{\mu}_n(\tilde{X}_n), \\ \mu_{n-1}(X_n[i]) &= \tilde{\mu}_{n-1}(\tilde{X}_n[i]), \quad i \in S_\beta, \end{aligned}$$

it follows that

$$W_n = \log_2 \left[\frac{\tilde{\mu}_n(\tilde{X}_n)}{\lambda_n(\tilde{X}_n)} \right]$$

almost surely $[P]$. Let $d\tilde{\mu}_n/d\lambda_n$ be the Radon-Nikodym derivative of $\tilde{\mu}_n$ with respect to λ_n . The function $\log_2(d\tilde{\mu}_n/d\lambda_n)$ is called the information density of $\tilde{\mu}_n$ with respect to λ_n . We have

$$\begin{aligned} E_P[W_n] &= \int_{V_0^{\beta^n}} \log_2(d\tilde{\mu}_n/d\lambda_n) d\tilde{\mu}_n, \\ E_P[|W_n|] &= \int_{V_0^{\beta^n}} |\log_2(d\tilde{\mu}_n/d\lambda_n)| d\tilde{\mu}_n. \end{aligned}$$

Inequality (51) then follows from the inequality

$$\int |\log_2(d\tilde{\mu}_n/d\lambda_n)| d\tilde{\mu}_n \leq 2e^{-1} \log_2 e + \int \log_2(d\tilde{\mu}_n/d\lambda_n) d\tilde{\mu}_n,$$

which is well known in statistics and information theory [3, Lemma 5.2.6].

Definition: Random variable h_μ for B-V source μ . Let $\mu \in \mathcal{S}(D)$, let $P = P_\mu$, and let Λ_P be the probability space $(\Omega_D, \mathcal{F}(\Omega_D), P)$. If $E \in \mathcal{F}(\Omega_D)$, the characteristic function of event E is the random variable $\chi_E : \Lambda_P \rightarrow \{0, 1\}$ such that $\{\chi_E = 1\} = E$. With W_n as defined in Lemma 5.1, a random variable h_μ is then defined on Λ_P such that

$$h_\mu = -(\beta - 1)^{-1} \left[\chi_{\{N=1\}} \log_2 \mu_1(X_1) + \sum_{n=2}^{\infty} \chi_{\{N=n\}} W_n \right] \quad (52)$$

holds almost surely $[P]$. Define

$$V_n(\mu) \triangleq \{x \in V_n : \mu(x) > 0\}, \quad n \geq 0.$$

If $\omega \in \Omega_D$, then $h_\mu(\omega)$ is evaluated as follows:

- If $N(\omega) = 1$ and $X_1(\omega) = x \in V_1(\mu)$, then

$$h_\mu(\omega) = -(\beta - 1)^{-1} \log_2 \mu(x). \quad (53)$$

- If $N(\omega) = n > 1$ and $X_n(\omega) = x \in V_n(\mu)$, then

$$h_\mu(\omega) = -(\beta - 1)^{-1} \log_2 \frac{\mu(x)}{\prod_{i=0}^{\beta-1} \mu(x[i])}. \quad (54)$$

Note that $\prod_{i=0}^{\beta-1} \mu(x[i]) > 0$ for $x \in V_n(\mu)$ by Lemma 5.1.

Lemma 5.2. Let $\mu \in \mathcal{S}(D)$, let $P = P_\mu$, and let Λ_P be the probability space $(\Omega_D, \mathcal{F}(\Omega_D), P)$. The random variable h_μ belongs $L^1[\Lambda_P]$ and its expected value is $H_\infty(\mu)$.

Proof. Since N is a stopping time relative to the random sequence $(Z_i : i \geq 0)$, the random variables $\chi_{\{N=n\}}$ and W_n on Λ_P are statistically independent for every $n \geq 1$. Thus,

$$\begin{aligned} E_P[\chi_{\{N=n\}} W_n] &= P\{N=n\} E_P[W_n] \\ &= (1 - \beta^{-1}) \beta^{-(n-1)} E_P[W_n] \\ &= (1 - \beta^{-1}) \beta^{-(n-1)} (\beta H(\mu_{n-1}) - H(\mu_n)) \end{aligned}$$

and

$$\begin{aligned} E_P[\chi_{\{N=n\}} |W_n|] &= P\{N=n\} E_P[|W_n|] \\ &= (1 - \beta^{-1}) \beta^{-(n-1)} E_P[|W_n|] \\ &\leq (1 - \beta^{-1}) \beta^{-(n-1)} (C + \beta H(\mu_{n-1}) - H(\mu_n)), \end{aligned}$$

where $C = 2e^{-1} \log_2 e$. From equation (52), we have

$$|h_\mu| = (\beta - 1)^{-1} \left[-\chi_{\{N=1\}} \log_2 \mu_1(X_1) + \sum_{n=2}^{\infty} \chi_{\{N=n\}} |W_n| \right].$$

By the monotone convergence theorem, it is legitimate to integrate the right side of this equation term by term. This gives us

$$\begin{aligned} E_P[|h_\mu|] &= \beta^{-1} \left[E_P[-\log_2 \mu_1(X_1)] + \sum_{n=2}^{\infty} \beta^{-(n-1)} E_P[|W_n|] \right] \leq \\ &\beta^{-1} \left[H(\mu_1) + \sum_{n=2}^{\infty} \beta^{-(n-1)} \{C + \beta H(\mu_{n-1}) - H(\mu_n)\} \right] = \\ &[\beta(\beta - 1)]^{-1} C + 2\beta^{-1} H(\mu_1) - H_\infty(\mu) < \infty. \end{aligned}$$

The random variable $|h_\mu|$ consequently belongs to $L^1[\Lambda_P]$ and thus so does h_μ . By the dominated convergence theorem, the right side of equation (52) can be integrated term by term, giving us

$$E_P[h_\mu] = \beta^{-1} \left[E_P[-\log_2 \mu_1(X_1)] - \sum_{n=2}^{\infty} \beta^{-(n-1)} E_P[W_n] \right] =$$

$$\beta^{-1} \left[H(\mu_1) - \sum_{n=2}^{\infty} \beta^{-(n-1)} \{ \beta H(\mu_{n-1}) - H(\mu_n) \} \right] =$$

$$\beta^{-1} [H(\mu_1) - (H(\mu_1) - \beta H_{\infty}(\mu))] = H_{\infty}(\mu)$$

Definition: Random variable h_{μ}^* for B-V source μ . Let $\mu \in \mathcal{S}(D)$, let $P = P_{\mu}$, and let Λ_P be the probability space $(\Omega_D, \mathcal{F}(\Omega_D), P)$. Define

$$h_{\mu}^* \triangleq E_P[h_{\mu} | \mathcal{F}_T],$$

the conditional expectation of h_{μ} with respect to the sigma-field \mathcal{F}_T of T -invariant events. Since h_{μ} belongs to $L^1[\Lambda_P]$, h_{μ}^* also belongs to $L^1[\Lambda_P]$ and we have

$$E_P[h_{\mu}^*] = E_P[h_{\mu}] = H_{\infty}(\mu).$$

Theorem 5.3 (SMB Theorem). Let $D = (V, E)$ be a β -regular Bratteli diagram, where $\beta \geq 2$. Let $\mu \in \mathcal{S}(D)$, let $P = P_{\mu}$, and let Λ_P be the probability space $(\Omega_D, \mathcal{F}(\Omega_D), P)$. The sequence $\{-\beta^{-n} \log_2 \mu_n(X_n) : n \geq 0\}$ converges to h_{μ}^* in $L^1[\Lambda_P]$ norm. It also converges to h_{μ}^* almost surely $[P]$.

The proof of the SMB theorem is deferred to subsection V-A. The following result follows from Theorem 5.3 and is the asymptotic equipartition property for ergodic Bratteli-Vershik sources.

Asymptotic Equipartition Property. Let $D = (V, E)$ be a β -regular Bratteli diagram, where $\beta \geq 2$. Let $\mu \in \mathcal{S}_e(D)$, let $P = P_{\mu}$, and let Λ_P be the probability space $(\Omega_D, \mathcal{F}(\Omega_D), P)$. Then

$$\lim_{n \rightarrow \infty} E_P[|-\beta^{-n} \log_2 \mu_n(X_n) - H_{\infty}(\mu)|] = 0. \quad (55)$$

Also, for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} P\{|-\beta^{-n} \log_2 \mu_n(X_n) - H_{\infty}(\mu)| > \varepsilon\} = 0. \quad (56)$$

Proof. Since $P \in \mathcal{P}_e(\Omega_D, T)$, every \mathcal{F}_T -measurable random variable on Λ_P is constant almost surely $[P]$. Thus,

$$h_{\mu}^* = E_P[h_{\mu}^*] = H_{\infty}(\mu) \text{ a.s. } [P].$$

Statement (55) then follows from the convergence in $L^1[\Lambda_P]$ norm in the SMB theorem. The Markov inequality gives us

$$P\{|-\beta^{-n} \log_2 \mu_n(X_n) - H_{\infty}(\mu)| > \varepsilon\} \leq$$

$$\varepsilon^{-1} E_P[|-\beta^{-n} \log_2 \mu_n(X_n) - H_{\infty}(\mu)|],$$

and then statement (56) follows from (55).

Remark. From statement (56), it follows that

$$\lim_{n \rightarrow \infty} \mu_n(\{x \in V_n : 2^{-\beta^n(H_{\infty}(\mu) + \varepsilon)} \leq \mu_n(x) \leq 2^{-\beta^n(H_{\infty}(\mu) - \varepsilon)}\}) = 0$$

holds for every $\varepsilon > 0$. This tells us that for large n , μ_n is approximately a uniform distribution on a subset of V_n of cardinality roughly equal to $2^{\beta^n H_\infty(\mu)}$. This fact has implications for fixed-length lossy encoding of ergodic Bratteli-Vershik sources, as we shall see in Section VI.

The following result identifies the probability distribution of the SMB limit function h_μ^* in Theorem 5.3. In this result and subsequently, $\mathcal{F}(\mathbb{R})$ denotes the sigma-field of Borel subsets of \mathbb{R} .

Theorem 5.4. Let $D = (V, E)$ be a β -regular Bratteli diagram, where $\beta \geq 2$. Let $\mu \in \mathcal{S}(D)$, let $P = P_\mu$, and let Λ_P be the probability space $(\Omega_D, \mathcal{F}(\Omega_D), P)$. Then

$$P\{h_\mu^* \in G\} = \lambda_\mu(\{\sigma \in \mathcal{S}_e(D) : H_\infty(\sigma) \in G\}), \quad G \in \mathcal{F}(\mathbb{R}). \quad (57)$$

Remarks.

- For $\mu \in \mathcal{S}(D)$, let $F_\mu : \mathbb{R} \rightarrow [0, 1]$ be the cumulative distribution function defined in (6). Since L^1 convergence implies convergence in distribution, Theorems 5.3-5.4 give us

$$\lim_{n \rightarrow \infty} \mu_n(\{x \in V_n(\mu) : -\beta^{-n} \log_2 \mu_n(x) \leq h\}) = F_\mu(h)$$

at every point $h \in \mathbb{R}$ at which F_μ is continuous. This is the weak form of the SMB theorem (Theorem 1.9 of Section I).

- Formula (57) is easily seen to be true if $\mu \in \mathcal{S}_e(D)$, as follows. Using the fact that $h_\mu^* = H_\infty(\mu)$ almost surely $[P_\mu]$,

$$P_\mu\{h_\mu^* \in G\} = P_\mu(\{\omega \in \Omega_D : H_\infty(\mu) \in G\}),$$

which is 1 or 0 depending on whether $H_\infty(\mu)$ belongs to G or not. λ_μ is the point mass at μ , so that $\lambda_\mu(F)$ for an event $F \in \mathcal{F}(\mathcal{S}_e(D))$ is 1 or 0 depending on whether $\mu \in F$ or not. Thus, (57) clearly holds. However, the proof of (57) for $\mu \notin \mathcal{S}_e(D)$ is more involved, as we shall see below.

- An analogue of Theorem 5.4 is known for stationary sequential sources [34] or, more generally, for asymptotically mean stationary sources [3, Theorem 3.1.1].

Proof of Theorem 5.4. Fix $\mu \in \mathcal{S}(D)$. Let $P = P_\mu$ and we have the probability space $\Lambda_P = (\Omega_D, \mathcal{F}(\Omega_D), P)$. Note that $\sigma_n(V_n(\sigma)) = 1$ for each $\sigma \in \mathcal{S}(D)$. Let S_μ be the measurable subset of $\mathcal{S}_e(D)$ consisting of all $\sigma \in \mathcal{S}_e(D)$ such that σ_n is absolutely continuous with respect to μ_n for every $n \geq 0$. That is, $\sigma \in S_\mu$ if and only if

$$V_n(\sigma) = V_n(\sigma) \cap V_n(\mu), \quad n \geq 0.$$

By the ergodic decomposition theorem, $\lambda_\mu(S_\mu) = 1$. For $n \geq 0$, let $\phi_n : S_\mu \rightarrow [0, \infty)$ be the mapping

$$\phi_n(\sigma) \triangleq \beta^{-n} \sum_{x \in V_n(\sigma)} \sigma_n(x) \log_2 \frac{\sigma_n(x)}{\mu_n(x)}, \quad \sigma \in S_\mu.$$

($\beta^n \phi_n(\sigma)$ is simply the Kullback-Leibler distance between PMF σ_n and PMF μ_n and is therefore non-negative.) It is easy to verify that

$$\int_{S_\mu} \phi_n d\lambda_\mu = \beta^{-n} H_n(\mu) - \int_{S_\mu} \beta^{-n} H_n(\sigma) d\lambda_\mu(\sigma), \quad n \geq 0.$$

By the dominated convergence theorem and the ergodic decomposition of entropy rate, we then have

$$\lim_{n \rightarrow \infty} \int_{S_\mu} \phi_n d\lambda_\mu = H_\infty(\mu) - \int_{S_\mu} H_\infty(\sigma) d\lambda_\mu(\sigma) = 0.$$

By a well-known property of Kullback-Leibler distance [3, Lemma 5.2.6], for every $n \geq 0$ we have

$$\sum_{x \in V_n(\sigma)} \sigma_n(x) \left| \log_2 \frac{\sigma_n(x)}{\mu_n(x)} \right| \leq (2e^{-1} \log_2 e) + \beta^n \phi_n(\sigma), \quad \sigma \in S_\mu.$$

For $\sigma \in S_\mu$, $\varepsilon > 0$, and $n \geq 0$, we define subsets of V_n by

$$\begin{aligned} W_1(\sigma, \varepsilon, n) &\triangleq \{x \in V_n(\sigma) : \left| \beta^{-n} \log_2 \frac{\sigma_n(x)}{\mu_n(x)} \right| > \varepsilon\} \\ W_2(\sigma, \varepsilon, n) &\triangleq \{x \in V_n(\sigma) : |-\beta^{-n} \log_2 \sigma_n(x) - H_\infty(\sigma)| > \varepsilon\} \\ W(\sigma, \varepsilon, n) &\triangleq W_1(\sigma, \varepsilon, n) \cup W_2(\sigma, \varepsilon, n) \end{aligned}$$

We have

$$\sigma_n(W_1(\sigma, \varepsilon, n)) \leq \varepsilon^{-1} [(2e^{-1} \log_2 e) \beta^{-n} + \phi_n(\sigma)],$$

and hence

$$\lim_{n \rightarrow \infty} \int_{S_\mu} \sigma_n(W_1(\sigma, \varepsilon, n)) d\lambda_\mu(\sigma) = 0, \quad \varepsilon > 0. \quad (58)$$

We also have

$$\lim_{n \rightarrow \infty} \int_{S_\mu} \sigma_n(W_2(\sigma, \varepsilon, n)) d\lambda_\mu(\sigma) = 0, \quad \varepsilon > 0. \quad (59)$$

(The integrand converges to zero pointwise for $\sigma \in S_\mu$ by the asymptotic equipartition property, and then integrate, applying the dominated convergence theorem.) We conclude from (58)-(59) that

$$\lim_{n \rightarrow \infty} \int_{S_\mu} \sigma_n(W(\sigma, \varepsilon, n)) d\lambda_\mu(\sigma) = 0, \quad \varepsilon > 0. \quad (60)$$

In the following, fix $y \in \mathbb{R}$ and $\varepsilon > 0$. For $\sigma \in S_\mu$ and $n \geq 0$, we have

$$\begin{aligned} \{x \in V_n(\sigma) : -\beta^{-n} \log_2 \mu_n(x) \leq y\} &\subset \\ \{x \in V_n(\sigma) : H_\infty(\sigma) \leq y + 2\varepsilon\} &\cup W(\sigma, \varepsilon, n), \end{aligned}$$

and therefore

$$\sigma_n(\{x \in V_n : -\beta^{-n} \log_2 \mu_n(x) \leq y\}) \leq$$

$$\begin{aligned} \sigma_n(\{x \in V_n : H_\infty(\sigma) \leq y + 2\varepsilon\}) + \sigma_n(W(\sigma, \varepsilon, n)) = \\ G(\sigma) + \sigma_n(W(\sigma, \varepsilon, n)), \end{aligned}$$

where $G : S_\mu \rightarrow \mathbb{R}$ is the function such that

$$G(\sigma) \triangleq \begin{cases} 1, & H_\infty(\sigma) \leq y + 2\varepsilon \\ 0, & \text{otherwise} \end{cases}$$

Integrating,

$$\begin{aligned} \mu_n(\{x \in V_n : -\beta^{-n} \log_2 \mu_n(x) \leq y\}) = \\ \int_{S_\mu} \sigma_n(\{x \in V_n : -\beta^{-n} \log_2 \mu_n(x) \leq y\}) d\lambda_\mu(\sigma) \leq \\ \int_{S_\mu} G(\sigma) d\lambda_\mu(\sigma) + \int_{S_\mu} \sigma_n(W(\sigma, \varepsilon, n)) d\lambda_\mu(\sigma). \end{aligned}$$

Using (60) and the fact that

$$\int_{S_\mu} G(\sigma) d\lambda_\mu(\sigma) = \lambda_\mu(\{\sigma \in \mathcal{S}_e(D) : H_\infty(\sigma) \leq y + 2\varepsilon\}),$$

we obtain the upper bound

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \mu_n(\{x \in V_n : -\beta^{-n} \log_2 \mu_n(x) \leq y\}) \leq \\ \lambda_\mu(\{\sigma \in \mathcal{S}_e(D) : H_\infty(\sigma) \leq y + 2\varepsilon\}). \end{aligned} \quad (61)$$

Note that

$$\begin{aligned} \{x \in V_n(\sigma) : H_\infty(\sigma) \leq y - 2\varepsilon\} \subset \\ \{x \in V_n(\sigma) : -\beta^{-n} \log_2 \mu_n(x) \leq y\} \cup W(\sigma, \varepsilon, n). \end{aligned}$$

We obtain from this the lower bound

$$\begin{aligned} \underline{\lim}_{n \rightarrow \infty} \mu_n(\{x \in V_n : -\beta^{-n} \log_2 \mu_n(x) \leq y\}) \geq \\ \lambda_\mu(\{\sigma \in \mathcal{S}_e(D) : H_\infty(\sigma) \leq y - 2\varepsilon\}), \end{aligned} \quad (62)$$

via reasoning steps similar to the steps used in obtaining (61). Let $F_1 : \mathbb{R} \rightarrow [0, 1]$ and $F_2 : \mathbb{R} \rightarrow [0, 1]$ be the cumulative distribution functions defined by

$$\begin{aligned} F_1(y) &\triangleq P\{h_\mu^* \leq y\}, \quad y \in \mathbb{R} \\ F_2(y) &\triangleq \lambda_\mu(\{\sigma \in \mathcal{S}_e(D) : H_\infty(\sigma) \leq y\}), \quad y \in \mathbb{R} \end{aligned}$$

We complete the proof by showing that $F_1 = F_2$, which implies statement (57). Let Q be the set of all real numbers at which F_1, F_2 are both continuous. Let $y \in Q$. By almost sure convergence in the SMB theorem, we also have convergence in distribution. Thus,

$$\lim_{n \rightarrow \infty} \mu_n(\{x \in V_n : -\beta^{-n} \log_2 \mu_n(x) \leq y\}) = F_1(y).$$

By (61)-(62),

$$F_2(y - 2\varepsilon) \leq F_1(y) \leq F_2(y + 2\varepsilon), \quad \varepsilon > 0.$$

Letting $\varepsilon \rightarrow 0$, we obtain

$$F_2(y) \leq F_1(y) \leq F_2(y)$$

and hence

$$F_1(y) = F_2(y), \quad y \in Q.$$

The complement of Q is a countable set and therefore Q is dense in \mathbb{R} . Fix any real number y . Our proof is complete once we show $F_1(y) = F_2(y)$. Let $\{y_n : n \geq 1\}$ be a sequence in Q converging downward to y . We have

$$F_1(y_n) = F_2(y_n), \quad n \geq 1,$$

and F_1 and F_2 are right continuous functions. Therefore,

$$F_1(y) = \lim_{n \rightarrow \infty} F_1(y_n) = \lim_{n \rightarrow \infty} F_2(y_n) = F_2(y).$$

5.1 Proof of Theorem 5.3

Fix β -regular Bratteli diagram $D = (V, E)$ and $\mu \in \mathcal{S}(D)$. Let $P = P_\mu$. We have the probability space $\Lambda_P = (\Omega_D, \mathcal{F}(\Omega_D), P)$. Define random variable $U_i = h_\mu \circ T^i$ on Λ_P ($i \in \mathbb{Z}$). The following result plays a key role in our proof of Theorem 5.3.

Lemma 5.5. For $n \geq 1$,

$$-\log_2 \mu_n(X_n) = \sum_{j=-Z^{(n)}}^{-Z^{(n)} + \beta^n - 2} U_j \quad \text{a.s. } [P]. \quad (63)$$

Proof. Fix $n \geq 1$ and $x \in V_n$ such that $\mu_n(x) > 0$. For each $i \in S_{\beta, n} = \{0, 1, \dots, \beta^n - 1\}$, recall that we have the cylinder subset

$$C_n(i, x) = \{X_n = x, Z^{(n)} = i\}$$

of Ω_D . We show that

$$-\log_2 \mu_n(x) = \sum_{j=-i}^{-i + \beta^n - 2} h_\mu(T^j \omega), \quad \omega \in C_n(i, x), \quad i \in S_{\beta, n}, \quad (64)$$

which establishes the statement (63). Let $t(n)$ be the finite rooted labeled tree in which

- Each non-leaf vertex u has β ordered child vertices, denoted by $c[u, 0], c[u, 1], \dots, c[u, \beta - 1]$.
- There are β^n leaves, and every root-to-leaf path consists of n edges.
- Each vertex u of $t(n)$ carries a label $x^u \in V$, determined as follows: the root vertex of $t(n)$ carries the label x , and for each non-leaf vertex u of $t(n)$, child vertex $w = c[u, i]$ carries label $x^w = x^u[i]$ ($i = 0, 1, \dots, \beta - 1$).

Let u^* denote the root vertex of $t(n)$. Let u be a leaf of $t(n)$. Starting at u and following the leaf-to-root path, we encounter vertices u_0, u_1, \dots, u_n in order, where $u_0 = u$, $u_n = u^*$, and u_j is the parent vertex of u_{j-1} ($j = 1, \dots, n$). The sequence $(i_0, i_1, \dots, i_{n-1}) \in \{0, 1, \dots, \beta - 1\}^n$ such that $u_j = c[u_{j+1}, i_j]$ ($j = 0, \dots, n-1$) is defined to be the *address* of u . Also, the integer

$$i_0 + i_1\beta + \dots + i_{n-1}\beta^{n-1}$$

is called the *index* of u . For each integer $i \in S_{\beta,n}$, we let $u(i)$ be the leaf of $t(n)$ with index i . The list $u(0), u(1), \dots, u(\beta^n - 1)$ provides an enumeration of all leaves of $t(n)$. Let $\mathcal{V}(t(n))$ be the set of all vertices of $t(n)$ and let $\mathcal{V}^+(t(n))$ denote the set of all non-leaf vertices of $t(n)$. It is easy to verify the following property:

- **Property 1:** Let $\phi : \mathcal{V}(t(n)) \rightarrow \mathbb{R}$ be any function such that $\phi(u) = 0$ for every leaf u of $t(n)$. Let $\Psi_\phi : \mathcal{V}^+(t(n)) \rightarrow \mathbb{R}$ be the function

$$\Psi_\phi(u) \triangleq \phi(u) - \sum_{i=0}^{\beta-1} \phi(c[u, i]), \quad u \in \mathcal{V}^+(t(n)).$$

Then

$$\phi(u^*) = \sum_{u \in \mathcal{V}^+(t(n))} \Psi_\phi(u).$$

Let

$$\mathcal{V}_1(t(n)) = \{u(i) : 0 \leq i < \beta^n - 1\}.$$

Note that $\mathcal{V}_1(t(n))$ is the set of all leaves of $t(n)$ other than the leaf $u(\beta^n - 1)$, which is the only leaf of $t(n)$ whose address has every entry equal to $\beta - 1$. We define mapping $F : \mathcal{V}_1(t(n)) \rightarrow \mathcal{V}^+(t(n))$ as follows. Let $u \in \mathcal{V}_1(t(n))$, and let (i_0, \dots, i_{n-1}) be the address of u . Let J be the smallest integer $j \in \{0, \dots, n-1\}$ such that $i_j < \beta - 1$. Letting u_0, \dots, u_n be the vertices in order along to leaf-to-root path starting at u , we define $F(u) = u_{j+1}$. The function F obeys the following property:

- **Property 2:** F maps onto $\mathcal{V}^+(t(n))$. Moreover,

$$|\{u \in \mathcal{V}_1(t(n)) : F(u) = w\}| = \beta - 1, \quad w \in \mathcal{V}^+(t(n)).$$

(For example, the $\beta - 1$ leaves in $\mathcal{V}_1(t(n))$ mapped into u^* by F have the addresses $(\beta - 1, \beta - 1, \dots, \beta - 1, i)$ ($i = 0, 1, \dots, \beta - 2$), from which it can be worked out that this set of leaves is $\{u(\beta^{n-1}i - 1) : i = 1, \dots, \beta - 1\}$.) Letting ϕ be a function under Property 1, then, combining Properties 1-2, we conclude

$$\phi(u^*) = (\beta - 1)^{-1} \sum_{w \in \mathcal{V}_1(t(n))} \Psi_\phi(F(w)). \quad (65)$$

By choosing an appropriate function ϕ in equation (65), we will obtain (64), completing our proof. By Lemma 5.1,

$$\mu(x^u) > 0, \quad u \in \mathcal{V}(t(n)).$$

This property ensures the existence of the particular function $\phi : \mathcal{V}(t(n)) \rightarrow \mathbb{R}$ which is 0 over the leaves of $t(n)$ and satisfies

$$\phi(u) = -\log_2 \mu(x^u), \quad u \in \mathcal{V}^+(t(n)).$$

Let $\omega \in C_n(k, x)$, where $k < \beta^n - 1$. The path $\omega_0^{n-1} \in \Pi_D(0, n, x)$ and the leaf $u(k)$ of $t(n)$ have the same address $(i_0, \dots, i_{n-1}) = [k]_{\beta, n}$. Letting u_0, \dots, u_n be the vertices of $t(n)$ along to leaf-to-root path starting at $u(k)$, the label assigned to vertex u_j is $X_j(\omega)$. Let $N = N(\omega)$. Letting J be the smallest integer $j \in \{0, \dots, n-1\}$ such that $i_j < \beta - 1$, we have $J+1 = N$, and therefore $F(u(k)) = u_N$ and $x^{u_N} = X_N(\omega)$. If $N = 1$, the children of u_N are leaves of $t(n)$, and we have

$$\Psi_\phi(F(u(k))) = \phi(u_N) - \sum_{i=0}^{\beta-1} \phi(c[u_N, i]) =$$

$$\phi(u_N) = -\log \mu(X_1(\omega)) = (\beta - 1)h_\mu(\omega)$$

by formula (53). If $N > 1$, the children of u_N are not leaves of $t(n)$, and we have

$$\Psi_\phi(F(u(k))) = \phi(u_N) - \sum_{i=0}^{\beta-1} \phi(c[u_N, i]) =$$

$$-\log \mu(X_N(\omega)) + \sum_{i=0}^{\beta-1} \log_2 \mu(X_N(\omega)[i]) = (\beta - 1)h_\mu(\omega)$$

by formula (54). We have shown that

$$\Psi_\phi(F(u(k))) = (\beta - 1)h_\mu(\omega), \quad \omega \in C_n(k, x), \quad 0 \leq k \leq \beta^n - 2. \quad (66)$$

Now let $i \in S_{\beta, n}$ be arbitrary and let $\omega \in C_n(i, x)$. Then a property of the Vershik transformation tells us that

$$T^j \omega \in C_n(i + j, x), \quad -i \leq j \leq -i + \beta^n - 1.$$

Combining this statement with (66), we have

$$\Psi_\phi(F(u(i + j))) = (\beta - 1)h_\mu(T^j \omega), \quad -i \leq j \leq -i + \beta^n - 2,$$

and also

$$\{u(i + j) : -i \leq j \leq -i + \beta^n - 2\} = \mathcal{V}_1(t(n)).$$

Employing (65), we have

$$(\beta - 1) \sum_{j=-i}^{-i+\beta^n-2} h_\mu(T^j \omega) = \sum_{w \in \mathcal{V}_1(t(n))} \Psi_\phi(F(w)) =$$

$$(\beta - 1)\phi(u^*) = -(\beta - 1)\log_2 \mu_n(x),$$

and thus formula (64) holds, completing our proof.

Proof of almost sure convergence in Theorem 5.3. By Lemma 5.5, for each $n \geq 1$, we have

$$-\log_2 \mu_n(X_n) = \left(\sum_{j=-Z^{(n)}}^0 U_j \right) + \left(\sum_{j=-1}^{-Z^{(n)}+\beta^n-2} U_j \right) - U_{-1} - U_0$$

almost surely $[P]$. Both of the sums on the right side of the preceding equation are non-empty, consisting of $Z^{(n)} + 1$ terms and $\beta^n - Z^{(n)}$ terms, respectively. (Since $0 \leq Z^{(n)} \leq \beta^n - 1$, the random variables $Z^{(n)} + 1$ and $\beta^n - Z^{(n)}$ are positive integer valued.) We then have

$$\begin{aligned} -\beta^{-n} \log_2 \mu_n(X_n) &= \left(\frac{Z^{(n)} + 1}{\beta^n} \right) \left(\frac{\sum_{j=-Z^{(n)}}^0 U_j}{Z^{(n)} + 1} \right) \\ &+ \left(\frac{\beta^n - Z^{(n)}}{\beta^n} \right) \left(\frac{\sum_{j=-1}^{-Z^{(n)}+\beta^n-2} U_j}{\beta^n - Z^{(n)}} \right) - \frac{U_{-1} + U_0}{\beta^n} \end{aligned} \quad (67)$$

Note that

$$Z^{(n+1)} - Z^{(n)} = \beta^n Z_n \geq 0.$$

Since the sequence (Z_0, Z_1, \dots) is aperiodic, Z_n must be > 0 for infinitely many n . It follows that $Z^{(n)} \rightarrow \infty$ everywhere on Ω_D . Thus,

$$\lim_{n \rightarrow \infty} \left(\frac{\sum_{j=-Z^{(n)}}^0 U_j}{Z^{(n)} + 1} \right) = h_\mu^* \text{ a.s. } [P], \quad (68)$$

because the pointwise ergodic theorem [35, Thm. 3.3.6] tells us that

$$\lim_{m \rightarrow \infty} (m+1)^{-1} \sum_{j=-m}^0 U_j = h_\mu^* \text{ a.s. } [P].$$

Also, note that

$$(\beta^{n+1} - Z^{(n+1)}) - (\beta^n - Z^{(n)}) = \beta^n[(\beta - 1) - Z_n] \geq 0.$$

We must have $Z_n < \beta - 1$ for infinitely many n , so $\beta^n - Z^{(n)} \rightarrow \infty$ everywhere on Ω_D . Thus,

$$\lim_{n \rightarrow \infty} \left(\frac{\sum_{j=-1}^{-Z^{(n)}+\beta^n-2} U_j}{\beta^n - Z^{(n)}} \right) = h_\mu^* \text{ a.s. } [P], \quad (69)$$

because the pointwise ergodic theorem tells us that

$$\lim_{m \rightarrow \infty} m^{-1} \sum_{j=-1}^{m-2} U_j = h_\mu^* \text{ a.s. } [P].$$

Statements (67)-(69) easily imply that

$$\lim_{n \rightarrow \infty} -\beta^{-n} \log_2 \mu_n(X_n) = h_\mu^* \text{ a.s. } [P].$$

Proof of $L^1[\Lambda_P]$ convergence in Theorem 5.3. Via equation (63) and the fact that $0 \leq Z^{(n)} \leq \beta^n - 1$,

$$|-\beta^{-n} \log_2 \mu_n(X_n)| \leq 2 \left[\frac{\sum_{j=-\beta^n}^{\beta^n-1} |U_j|}{2\beta^n} \right] \quad (70)$$

holds almost surely for each $n \geq 1$. By the L^1 ergodic theorem [35, Thm. 3.3.7], the sequence $\{(2m)^{-1} \sum_{j=-m}^{m-1} |U_j| : m \geq 1\}$ is convergent in $L^1[\Lambda_P]$ norm; therefore, the sequence is uniformly integrable. Consequently, appealing to the bound (70), the sequence $\{-\beta^{-n} \log_2 \mu_n(X_n) : n \geq 0\}$ is also uniformly integrable. A uniformly integrable sequence of random variables which is almost surely convergent is also convergent in L^1 norm (to the same limit function) [33, Cor. 6.5.5]. Thus, the sequence $\{-\beta^{-n} \log_2 \mu_n(X_n) : n \geq 0\}$ converges in $L^1[\Lambda_P]$ norm to h_μ^* .

6 Fixed-Length Lossy Source Encoding

Let $D = (V, E)$ be a β -regular Bratteli diagram. Let $n \geq 0$. We define an n -th order fixed-length encoder for D to be any mapping $\phi_n : V_n \rightarrow \{0, 1\}^*$ in which all of the binary codewords in $\phi_n(V_n)$ have the same length. We do not require that ϕ_n be one-to-one, that is, we are now allowing lossy encoding rather than lossless encoding. The encoding rate $R(\phi_n)$ of n -th order fixed-length encoder ϕ_n is defined to be L_n/β^n , where L_n is the fixed length of ϕ_n 's codewords.

Let $\delta \in (0, 1)$. Let μ be any source in $\mathcal{S}(D)$. We define $\mathbb{E}_\mu(\delta)$ to be the set of all sequences $\{\phi_n : n \geq 0\}$ such that

- **(a):** For each $n \geq 0$, ϕ_n is an n -th order fixed-length encoder for D .
- **(b):** For each $n \geq 0$, there is a mapping $\psi_n : \phi_n(V_n) \rightarrow V_n$ such that

$$\mu_n(\{x \in V_n : \psi_n(\phi_n(x)) = x\}) \geq 1 - \delta. \quad (71)$$

The sequences in $\mathbb{E}_\mu(\delta)$ shall be referred to as δ -lossy encoder sequences for μ . The parameter δ is called the *error level* of the encoders in $\mathbb{E}_\mu(\delta)$. The error level δ controls the degree to which an encoder sequence $\{\phi_n\}$ can be lossy; with the δ -lossy requirement, we are stipulating that a vertex in V_n can be decoded from its binary codeword except for a set of vertices of μ_n -probability at most δ ($n \geq 0$).

Definition. For a given error level $\delta \in (0, 1)$ and source $\mu \in \mathcal{S}(D)$, we wish to examine how small the rate sequence $\{R(\phi_n) : n \geq 0\}$ can become asymptotically as $n \rightarrow \infty$ for the encoder sequences $\{\phi_n\}$ in $\mathbb{E}_\mu(\delta)$. We make this idea precise as follows. We define source μ to be *stably encodable* at error level δ if there exists a (necessarily unique) non-negative real number $R^*(\delta, \mu)$ such that both of the following two statements hold:

- **(a):** There exists an encoder sequence $\{\phi_n\}$ in $\mathbb{E}_\mu(\delta)$ for which

$$\overline{\lim}_{n \rightarrow \infty} R(\phi_n) \leq R^*(\delta, \mu). \quad (72)$$

- **(b):** For any encoder sequence in $\{\phi_n\}$ in $\mathbb{E}_\mu(\delta)$,

$$\lim_{n \rightarrow \infty} R(\phi_n) \geq R^*(\delta, \mu). \quad (73)$$

A question of interest is to determine whether a given Bratteli-Vershik source $\mu \in \mathcal{S}(D)$ is stably encodable at a given error level δ , and, if so, to specify how $R^*(\delta, \mu)$ is to be computed. Theorem 6.1 below elucidates this question. It implies that μ is stably encodable at certain error levels related to the SMB theorem limit function, and computes $R^*(\delta, \mu)$ at these levels δ . The exceptional error levels not covered by Theorem 6.1 are at most countable in number.

Definition. Let $\mu \in \mathcal{S}(D)$. Let $F_\mu : \mathbb{R} \rightarrow [0, 1]$ be the cumulative distribution function defined in (6). For each $\delta \in (0, 1)$, we define

$$\begin{aligned} R^+(\delta, \mu) &\triangleq \inf\{x \in \mathbb{R} : F_\mu(x) > 1 - \delta\} \\ &= \inf\{x \in \mathbb{R} : \lambda_\mu(\{\sigma \in \mathcal{S}_e(D) : H_\infty(\sigma) \leq x\}) > 1 - \delta\}. \end{aligned}$$

and we define

$$\begin{aligned} R^-(\delta, \mu) &\triangleq \sup\{x \in \mathbb{R} : F_\mu(x) < 1 - \delta\} \\ &= \sup\{x \in \mathbb{R} : \lambda_\mu(\{\sigma \in \mathcal{S}_e(D) : H_\infty(\sigma) < x\}) < 1 - \delta\}. \end{aligned}$$

Discussion. Fix $\mu \in \mathcal{S}(D)$ throughout this discussion. Both $R^-(\delta, \mu)$ and $R^+(\delta, \mu)$ are nonincreasing functions of $\delta \in (0, 1)$. We have

$$0 \leq R^-(\delta, \mu) \leq R^+(\delta, \mu), \quad 0 < \delta < 1.$$

Suppose we have a particular $\delta \in (0, 1)$ for which $R^-(\delta, \mu) < R^+(\delta, \mu)$. As discussed in [7] [36], we then have

$$\begin{aligned} \lambda_\mu(\{\sigma \in \mathcal{S}_e : H_\infty(\sigma) \leq R^-(\delta, \mu)\}) &= 1 - \delta \\ \lambda_\mu(\{\sigma \in \mathcal{S}_e : H_\infty(\sigma) \geq R^+(\delta, \mu)\}) &= \delta, \end{aligned}$$

which implies that the open interval $(R^-(\delta, \mu), R^+(\delta, \mu))$ yields an “entropy rate gap”, meaning that

$$\lambda_\mu(\{\sigma \in \mathcal{S}_e(D) : R^-(\delta, \mu) < H_\infty(\sigma) < R^+(\delta, \mu)\}) = 0.$$

There can be only countably many such entropy rate gaps since they are pairwise disjoint and each one contains a rational number. We conclude that

$$R^-(\delta, \mu) = R^+(\delta, \mu)$$

for all but countably many $\delta \in (0, 1)$.

The following result tells us about lossy fixed-length encoding of non-ergodic Bratteli sources. Parthasarathy [7] proved the analogous result for stationary non-ergodic finite-alphabet sequential sources.

Theorem 6.1. Let $\mu \in \mathcal{S}(D)$. Let $\delta \in (0, 1)$. Then the following statements hold.

- **(a):** There exists an encoder sequence $\{\phi_n\}$ in $\mathbb{E}_\mu(\delta)$ for which

$$\overline{\lim}_{n \rightarrow \infty} R(\phi_n) \leq R^+(\delta, \mu). \quad (74)$$

- **(b):** For any encoder sequence in $\{\phi_n\}$ in $\mathbb{E}_\mu(\delta)$,

$$\underline{\lim}_{n \rightarrow \infty} R(\phi_n) \geq R^-(\delta, \mu). \quad (75)$$

- **(c):** If $R^-(\delta, \mu) = R^+(\delta, \mu)$, then μ is stably encodable at error level δ and

$$R^*(\delta, \mu) = R^-(\delta, \mu).$$

- **(d):** If μ is stably encodable at error level δ , then

$$R^-(\delta, \mu) \leq R^*(\delta, \mu) \leq R^+(\delta, \mu).$$

Remarks.

- Let $M_n(\delta, \mu)$ be the integer defined by (7). It is straightforward to show that any encoding scheme $\{\phi_n\}$ in $\mathbb{E}_\mu(\delta)$ for which

$$R(\phi_n) = \beta^{-n} \lceil \log_2 M_n(\delta, \mu) \rceil, \quad n \geq 0$$

yields the minimum $R(\phi_n)$ for every $n \geq 0$. It follows that μ is stably encodable at error level δ if and only if $\lim_n \beta^{-n} \log_2 M_n(\delta, \mu)$ exists, in which case $R^*(\delta, \mu)$ equals this limit. From these observations, one sees that Theorem 1.10 is equivalent to Theorem 6.1.

- Let $\mu \in \mathcal{S}_e(D)$. Then μ is stably encodable at every error level, and

$$R^*(\delta, \mu) = H_\infty(\mu), \quad 0 < \delta < 1.$$

Equivalently,

$$\lim_{n \rightarrow \infty} \beta^{-n} \log_2 M_n(\delta, \mu) = H_\infty(\mu), \quad 0 < \delta < 1.$$

This follows from Theorem 6.1 because λ_μ is the point mass at $\{\mu\}$, which implies

$$H_\infty(\mu) = R^-(\mu, \delta) = R^+(\mu, \delta), \quad 0 < \delta < 1.$$

- We know how to find non-ergodic B-V sources μ for which the exceptional set

$$S(\mu) = \{\delta : R^-(\delta, \mu) < R^+(\delta, \mu)\}$$

is non-empty and μ is stably encodable at every error level in $S(\mu)$. We also know how to find non-ergodic B-V sources μ for which $S(\mu)$ is non-empty and μ fails to be stably encodable at every error level in $S(\mu)$. Thus, Theorem 6.1 is not sufficient to completely analyze non-ergodic B-V sources; more work is needed.

Parts (c) and (d) of Theorem 6.1 follow from part (a) and part (b). Parts (a) and (b) are proved in the following two subsections, which complete Section 6.

6.1 Proof of Theorem 6.1(a)

Fix μ and δ . Let $\varepsilon > 0$ be arbitrary. It suffices to find a sequence $\{\phi_n : n \geq 0\}$ in $\mathbb{E}_\mu(\delta)$ for which

$$\overline{\lim}_{n \rightarrow \infty} R(\phi_n) \leq R^+(\delta, \mu) + \varepsilon. \quad (76)$$

Choose $h \in (R^+(\delta, \mu), R^+(\delta, \mu) + \varepsilon)$ such that F_μ is continuous at h . By definition of $R^+(\delta, \mu)$,

$$F_\mu(h) > 1 - \delta.$$

By the weak form of the SMB theorem (Theorem 1.9),

$$\lim_{n \rightarrow \infty} \mu_n(\{x \in V_n : -\beta^{-n} \log_2 \mu_n(x) \leq h\}) = F_\mu(h).$$

Letting

$$S_n \triangleq \{x \in V_n : \mu_n(x) \geq 2^{-\beta^n h}\}, \quad n \geq 0,$$

we have

$$\lim_{n \rightarrow \infty} \mu_n(S_n) > 1 - \delta.$$

Pick non-negative integer N such that

$$\mu_n(S_n) > 1 - \delta, \quad n \geq N.$$

Note that

$$|S_n| \leq 2^{\beta^n h}, \quad n \geq 0.$$

For $0 \leq n < N$, let ϕ_n be any fixed-length n -th order lossless encoder. For $n \geq N$, let ϕ_n be a fixed-length n -th order encoder which is one-to-one when restricted to S_n and employs the codeword length $\lceil \log_2 |S_n| \rceil$. Then $\{\phi_n\}$ belongs to $\mathbb{E}_\mu(\delta)$ and (76) holds because

$$R(\phi_n) \leq \beta^{-n} \lceil \log_2 |S_n| \rceil \leq \beta^{-n} + h \leq \beta^{-n} + R^+(\delta, \mu) + \varepsilon, \quad n \geq N.$$

6.2 Proof of Theorem 6.1(b)

Let $\{\phi_n : n \geq 0\}$ be an encoder sequence in $\mathbb{E}_\mu(\delta)$. For each $n \geq 0$, we have a decoder mapping $\psi_n : \phi_n(V_n) \rightarrow V_n$ such that (71) holds. Define

$$F_n \triangleq \{x \in V_n : \psi_n(\phi_n(x)) = x\}, \quad n \geq 0.$$

We have

$$\mu_n(F_n) \geq 1 - \delta, \quad n \geq 0.$$

Let L_n be the codeword length employed by encoder ϕ_n . Then $\phi_n(F_n) \subset \{0, 1\}^{L_n}$, so we have

$$|\phi_n(F_n)| \leq 2^{L_n} = 2^{\beta^n R(\phi_n)}.$$

There is a one-to-one correspondence between the sets F_n and $\phi_n(F_n)$, and so these two sets have the same cardinality. Thus, we have

$$\log_2 |F_n| \leq \beta^n R(\phi_n), \quad n \geq 0.$$

Inequality (75) will follow once we show that

$$\liminf_{n \rightarrow \infty} \beta^{-n} \log_2 |F_n| \geq R^-(\delta, \mu). \quad (77)$$

If $R^-(\delta, \mu) = 0$, there is nothing to prove, so assume that $R^-(\delta, \mu) > 0$. Let ε be an arbitrary number in the interval $(0, R^-(\delta, \mu))$. Choose $h \in (R^-(\varepsilon, \mu) - \varepsilon, R^-(\delta, \mu))$ such that F_μ is continuous at h . By definition of $R^-(\delta, \mu)$,

$$F_\mu(h) < 1 - \delta.$$

By the weak form of the SMB theorem (Theorem 1.9),

$$\lim_{n \rightarrow \infty} \mu_n(\{x \in V_n : -\beta^{-n} \log_2 \mu_n(x) < h\}) = F_\mu(h).$$

Let $\delta' = 1 - F_\mu(h)$, and define

$$S_n \triangleq \{x \in V_n : \mu_n(x) \leq 2^{-\beta^n h}\}, \quad n \geq 0.$$

Then

$$\lim_{n \rightarrow \infty} \mu_n(S_n) = \delta' > \delta.$$

We have

$$\mu_n(S_n \cap F_n) \geq \mu_n(F_n) + \mu_n(S_n) - 1 \geq \mu_n(S_n) - \delta,$$

and hence

$$\liminf_{n \rightarrow \infty} \mu_n(S_n \cap F_n) \geq \delta' - \delta > 0.$$

We may pick non-negative integer N such that

$$\mu_n(S_n \cap F_n) > 0, \quad n \geq N.$$

The sequence $\{\mu_n(S_n \cap F_n) : n \geq N\}$ is bounded away from 0, and hence

$$\lim_{n \rightarrow \infty} \beta^{-n} \log_2 \mu_n(S_n \cap F_n) = 0.$$

By definition of the set S_n , we have

$$|S_n \cap F_n| \geq 2^{\beta^n h} \mu_n(S_n \cap F_n), \quad n \geq 0.$$

Thus,

$$\beta^{-n} \log_2 |S_n \cap F_n| \geq h + \beta^{-n} \log_2 \mu_n(S_n \cap F_n), \quad n \geq N.$$

The last term on the right side drops out as $n \rightarrow \infty$, and we thus have

$$\liminf_{n \rightarrow \infty} \beta^{-n} \log_2 |F_n| \geq \liminf_{n \rightarrow \infty} \beta^{-n} \log_2 |S_n \cap F_n| \geq h > R^-(\delta, \mu) - \varepsilon.$$

As the preceding holds for all $\varepsilon > 0$ sufficiently small, (77) holds and our proof is complete.

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